Topology I - Homework 5

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October 18th, 2016

Exercise 8.1

Show that a sequence is a universal net if and only if it is eventually constant. We first notice that given $\{x_n\}_{n\in\mathbb{N}}$ a sequence, since $\mathbb{N}$ is a directed set with the usual order $\leq$ and letting $X = \bigcup_{n\in\mathbb{N}} x_n$ (with the discrete topology), we have that $\phi : \mathbb{N} \longrightarrow X$ defined by $\phi(n) = x_n$ is a net.

$\Longrightarrow$) We will proceed by contrapositive. Suppose we have a sequence $\{x_n\}_{n\in\mathbb{N}}$ that is not eventually constant, we will build a set $A \subset X$ such that ϕ is neither eventually in A or A^c, meaning that ϕ is not universal. Given such a sequence, we can select an infinite (countable) subsequence $\{y_i\}_{i\in\mathbb{N}}$ with $y_j \neq y_{j+1}$ for $j \in \mathbb{N}$. Define $A = \bigcup_{i\in\mathbb{N}} y_{2i}$, we have $\bigcup_{n\in\mathbb{N}} y_{2i+1} \subset A^c$.

Now ϕ cannot be eventually in A since for any $n \in \mathbb{N}$ fixed we have that there exists $k \geq n$ with $y_k \in \{y_i\}_{i\in\mathbb{N}}$ (because $\{y_i\}_{i\in\mathbb{N}}$ is a countable subsequence) and for such k we have $\phi(2k+1) = y_{2k+1} \in A^c$ (thus $\phi(2k+1) \notin A$ for $2k+1 > n$).

However, ϕ cannot be eventually in A^c since for any $n \in \mathbb{N}$ fixed we have that there exists $k \geq n$ with $y_k \in \{y_i\}_{i\in\mathbb{N}}$ (again because $\{y_i\}_{i\in\mathbb{N}}$ is a countable subsequence) and for such k we have $\phi(2k) = y_{2k} \in A$ (thus $\phi(2k) \notin A^c$ for $2k > n$).

$\Longleftarrow$) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence eventually constant, that is, there exists a fixed $N \in \mathbb{N}$ such that for $n \geq N$ we have $x_n = x$ for x fixed. We prove that for every subset $A \subset X$ we have ϕ eventually in either A or A^c:

1. If $x \in A$, then ϕ is eventually in A since for $N \in \mathbb{N}$ we have that for every $n \geq N$ then $\phi(n) = x_n = x \in A$.

2. If $x \in A^c$, then ϕ is eventually in A^c since for $N \in \mathbb{N}$ we have that for every $n \geq N$ then $\phi(n) = x_n = x \in A^c$.

Exercise 8.3

Let $\phi : D \longrightarrow X$ be a net that is eventually in A and frequently in B for some $A, B \subset X$. Show that ϕ is frequently in $A \cap B$,that is, we want to see that for every $\alpha \in D$ there exists $\beta \in D$ such that $\alpha \leq \beta$ and $\phi(\beta) \in A \cap B$. For this, we notice that since ϕ is eventually in A, there is some fixed $\alpha_A \in D$ such that if $\beta \geq \alpha_A$ then $\phi(\beta) \in A$.

Let now $\alpha \in D$ be fixed:

1. Suppose $\alpha \geq \alpha_A$, since ϕ is frequently in B (applied to α), there exists $\beta_B \in D$ with $\beta_B \geq \alpha$ and $\phi(\beta_B) \in B$. Moreover $\beta_B \geq \alpha \geq \alpha_A$ and by the observation above we have $\phi(\beta_B) \in A$, thus $\phi(\beta_B) \in A \cap B$ as desired.

2. Suppose $\alpha \leq \alpha_A$, since ϕ is frequently in B (applied to α_A), there exists $\beta_B \in D$ with $\beta_B \geq \alpha_A \geq \alpha$ and $\phi(\beta_B) \in B$. Again by the observation above we have $\phi(\beta_B) \in A$, thus $\phi(\beta_B) \in A \cap B$ as desired.

Hence ϕ is frequently in $A \cap B$.

Exercise 9.2

1. Show that arbitrary product of Hausdorff spaces is Hausdorff: let $\{X_j\}_{j\in J}$ be a family of Hausdorff spaces, $X = \prod_{j\in J} X_j$. Let $x, y \in X$ with $x \neq y$, because of this we have that there exists $i \in J$ with $x(i) \neq y(i)$, $x(i), y(i) \in X_i$. Since X_i is Hausdorff, there exist $U_i, V_i \subset X_i$ opens with $x(i) \in U_i$, $y(i) \in V_i$ and $U_i \cap V_i = \emptyset$. Consider $W_{i,U_i}, W_{i,V_i} \subset X$ opens. We have that $x \in W_{i,U_i}$ and $y \in W_{i,V_i}$ by construction, and if $z \in W_{i,U_i} \cap W_{i,V_i}$ then $z(i) \in U_i \cap V_i = \emptyset$ hence $W_{i,U_i} \cap W_{i,V_i} = \emptyset$, as desired. Thus X is Hausdorff.

2. Show that arbitrary product of regular spaces is regular: let $\{X_j\}_{j\in J}$ be a family of regular spaces, $X = \prod_{j\in J} X_j$. We will use the characterization of regular spaces as the ones where the closed neighborhoods form a neighborhood basis at any given point: given a point and a basic open, we will find a closed subset inside said open. Let $x \in X$ with $x \in W_{j_1,\ldots,j_n,U_1,\ldots,U_n}$ a basic open, we have $x(j_i) \in U_i \subset X_i$ for $i = 1, \ldots, n$. Since X_i is regular for $i \in J$, there are closed $C_i \subset U_i$ with $x(j_i) \in C_i$. Since the projections $\pi_i : X \longrightarrow X_i$ are continuous for every $i \in J$, we have that $\pi_i^{-1}(C_i)$ is closed and by the above $x \in \pi_i^{-1}(C_i) \subset \pi_i^{-1}(U_i) = W_{j_i,U_i}$. Hence considering $C = \pi_i^{-1}(C_1) \cap \cdots \cap \pi_i^{-1}(C_n)$ which is a finite intersection of closed thus closed. Moreover we have $x \in C \subset W_{j_1,\ldots,j_n,U_1,\ldots,U_n}$ since $z \in C$ implies $z(j_i) \in C_i \subset U_i$ for $i = 1, \ldots, n$, thus $z \in W_{j_1,\ldots,j_n,U_1,\ldots,U_n}$. As desired, we found a closed inside our basic open, hence the closed sets form a neighborhood basis, thus X is regular.

Exercise 9.3

Let X be a topological space, $\Delta = \{(x,x) : x \in X\}$. Show that X is Hausdorff if and only if Δ is closed in $X \times X$.

$\Longrightarrow$) Let X be Hausdorff, we will prove that Δ^c is open, hence Δ is open. Let $(x,y) \in \Delta^c$, that is, $x, y \in X$ with $x \neq y$. Since X is Hausdorff, there are opens $U, V \subset X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$ (notice $U \times V = W_{1,2,U,V}$ by construction, we will use the former for commodity). Moreover, we have that $\Delta \cap U \times V = \emptyset$ since $(z,z) \in U \times V$ implies $z \in U \cap V = \emptyset$. Now $U \times V$ is an open in $X \times X$ and $U \times V \subset \Delta^c$, hence Δ^c open as desired.

$\Longleftarrow$) Let Δ be closed, we want to prove that X is Hausdorff. Let $x, y \in X$ with $x \neq y$, we then have $(x,y) \in \Delta^c$ which is open. Thus, there is a basic open, say $W_{1,2,U,V} \subset \Delta^c$ with $(x,y) \in W_{1,2,U,V}$ (notice that since the open must be a subset of Δ^c, it cannot be of the form $W_{1,U}$ because this would mean that for $x \in U$ we have $(x,x) \in W_{1,U}$ but $(x,x) \notin \Delta^c$, a contradiction. An analogous argument shows that the open cannot be of the form $W_{2,V}$). Now $x \in U$, $y \in V$ both open and $U \cap V = \emptyset$ since if $z \in U \cap V$ for some $z \in X$ we have $(z,z) \in W_{1,2,U,V}$ meaning $W_{1,2,U,V} \not\subseteq \Delta^c$, a contradiction. We found two opens separating x and y, hence X is Hausdorff.