Topology I - Homework 6

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Exercise 9.5

We consider $\mathbf{S}$ the real line with the right half open interval topology. We want to see that $D = \{(x, -x) \in \mathbf{S} \times \mathbf{S} : x \in \mathbb{R}\}$ is a discrete subspace of $\mathbf{S} \times \mathbf{S}$.

To do this, we will prove that every point $(x, -x) \in D$ for $x \in \mathbb{R}$ is open. Since we can write any subset $A \subset D$ as $A = \bigcup_{a \in A} a$, proving that a is open automatically means that A, a union of opens, is open. This means that the subspace topology on D is the discrete topology. Since the opens $W_{1,2,U,V}$ (with U, V opens in $\mathbf{S}$) form a basis of $\mathbf{S} \times \mathbf{S}$, and the opens in D are of the form $W \cap D$ where W is an open in $\mathbf{S} \times \mathbf{S}$, it is enough to prove that given $x \in \mathbb{R}$ there is an open as above (an element of the basis of the topology in $\mathbf{S} \times \mathbf{S}$) such that $\{(x, -x)\} = W_{1,2,U,V} \cap D$.

Consider $U = [x, x+1), V = [-x, -x+1) \subset \mathbf{S}$ both open. We obviously have that $(x, -x) \in W_{1,2,U,V} \cap D$. Now suppose we have $a, b \in \mathbb{R}$ with $(a, b) \in W_{1,2,U,V} \cap D$, since they are in D we have $b = -a$, thus we consider elements $(a, -a) \in W_{1,2,U,V}$. This means that $a \in [x, x+1)$ and $-a \in [-x, -x+1)$, in particular $a \geq x$ and $a \leq x$ (since $-a \geq -x$), thus $x = a$. This proves $\{(x, -x)\} = W_{1,2,U,V} \cap D$, and the desired result follows.

Exercise 10.1

Let X be a compact Hausdorff space, we show that X is metrizable if and only if X is second countable.

$\Rightarrow$) Suppose X is metrizable, since it is compact by hypothesis, we can apply Exercise 6.1 and obtain directly that X is second countable.

$\Leftarrow$) Since X is compact and Hausdorff, we can apply Exercise 6.3 and obtain that X is regular. Now we have that X is regular and second countable by hypothesis, then applying the Urysohn Metrization Theorem we obtain that X is metrizable.

Exercise 10.2

Let X be compact Hausdorff, $n \geq 1$ and $\{f_i : X \longrightarrow \mathbb{R}\}_{i=1}^n$ a finite family of continuous functions such that for each $x, y \in X$, $x \neq y$ there is $1 \leq i \leq n$ with $f_i(x) \neq f_i(y)$. We show that X is homeomorphic to a subspace of $\mathbb{R}^n$.

To do this, we consider the following function:

$$\begin{array}{ccccccccc} f & : & X & \longrightarrow & \mathbb{R}^n & & & & \\ & & x & \longmapsto & f(x) & : & \{1,\dots,n\} & \longrightarrow & \mathbb{R} \\ & & & & & & m & \longmapsto & f_m(x) \end{array}$$

which is continuous by pointwise continuity (f_i continuous for $1 \leq i \leq n$). Moreover, since when $x, y \in X$ are such that $x \neq y$ we have that there is $1 \leq i \leq n$ with $f_i(x) \neq f_i(y)$, this means that $f(x) \neq f(y)$, proving that f is injective. In particular, f is a bijection from X to $f(X)$. Since X is compact and $\mathbb{R}^n$ is Hausdorff (because it is metric), we can apply the Midnight Theorems and say that f is automatically a homeomorphism, and hence X is homeomorphic to $f(X) \subset \mathbb{R}^n$. Thus we embedded X in $\mathbb{R}^n$, as desired.