Topology I - Homework 7

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Exercise 10.4

We have X a completely regular topological space, $A, B \subset X$ disjoint with A closed, B compact. We want to find a continuous function $f : X \longrightarrow I$ such that $f(A) = 0$ and $f(B) = 1$. Notice that if instead we find a continuous function $g : X \longrightarrow I$ such that $g(A) = 1$ and $g(B) = 0$, we can build the f desired by $f(x) = g(1 - x)$, which is continuous because is the composition of continuous functions ($1 - x$ is continuous from I to I). This is what we will do.

Consider now a point $b \in B$, we have $b \notin A$. By the Lemma 10.14 of the class notes, there exists a neighborhood U_b of b and a function $g_b : X \longrightarrow I$ such that $g_b(U_b) = 0$ and $g(A) = 1$ (notice that in particular this means $U_b \cap A = \emptyset$). By this process, for every $b \in B$ we have such U_b and g_b, hence $\{U_b\}_{b \in B}$ is an open cover of B. By compactness, we can reduce this to a finite subcover, say the given by the points b_i for $i = 1, \ldots, n$. Then we have functions $f_{b_i} : X \longrightarrow I$ and opens U_{b_i} for $i = 1, \ldots, n$ that together cover B. We want to build a continuous function g that is 0 when all g_{b_i} are and that is 1 when all g_{b_i} are. For this, consider the composition:

$$g : X \xrightarrow{(g_{b_1}, \ldots, g_{b_n})} I^n \xrightarrow{\prod_{i=1}^n x_i} I,$$

we have in practice that $g(x) = \prod_{i=1}^n g_{b_i}(x)$. The first function is continuous by pointwise continuity, and the second is continuous because is the multiplication of the components. Since composition of continuous functions is continuous, g is continuous. Moreover:

$$\begin{cases} g(b) = 0 \text{ for any } b \in U_{b_i} \text{ for any } i = 1, \ldots, n, \\ g(a) = 1 \text{ for any } a \in A, \end{cases}$$

hence $g(A) = 1$ and $g(B) = 0$, as desired.