Topology I - Homework 8

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Exercise 11.1

Let X be a locally compact space and A a closed subspace of X. Show that A is locally compact.

Consider $x \in A$, since $x \in X$ there is $K \subset X$ a compact neighborhood of x, that is, there is $U \subset K$ open with $x \in U$. We now prove that $K \cap A$ is compact. For this, consider $K \cap A \subset \bigcup_{i \in \mathbb{N}} U_i$ with $U_i \subset X$ open for $i \in \mathbb{N}$, an open covering. Now we clearly have $K \subset (K \cap A) \cup A^c \subset \bigcup_{i \in \mathbb{N}} U_i \cup A^c$, and since A is closed we obtain A^c open and this is an open covering of K. Since K is compact, we must have $K \subset U_1 \cup \cdots \cup U_n \cup A^c$ (notice that we may have $K \subset A$, but this is not guaranteed, and adding A^c to the right hand union ensures this is always true) and then $K \cap A \subset U_1 \cup \cdots \cup U_n$, a finite subcover of the original one. This means that $K \cap A$ is compact. Moreover, in the subspace topology we have that $U \cap A$ is open in A because U is open in X and $x \in U \cap A \subset K \cap A$ because $U \subset K$. Hence, for $x \in A$ we found the desired compact neighborhood, meaning that A is locally compact, as desired.

Exercise 11.2

Let X be a locally compact space and $f : X \longrightarrow Y$ an open surjective continuous function. Show that Y is locally compact.

Consider $y \in Y$. By surjectivity, there is a point $x \in X$ with $f(x) = y$. Since X is locally compact, there exists K_x a compact neighborhood of x, that is, there exists an open $U_x \subset K_x$ with $x \in U_x$. Since f is open, we have that $U_y = f(U_x)$ is open. Since f is continuous, we have that $K_y = f(K_x)$ is compact. We now have that $y = f(x) \subset f(U_x) \subset f(K_x)$ as desired, meaning that for $y \in Y$ we found K_y compact neighborhood of y, that is, Y is locally compact as desired.

Exercise 12.1

1. Show that $\mathbb{R}^2$ cannot be written as a countable union of closed sets each of which has empty interior.

   Suppose that $\mathbb{R}^2 = \bigcup_{i \in \mathbb{N}} A_i$ with A_i closed with empty interior for all $i \in \mathbb{N}$. Notice that since $\mathbb{R}^2$ is a locally compact Hausdorff space, it is a Baire space. This means that (the union of any countable family of closed subsets with empty interior has empty interior):

   $$\emptyset \neq \mathbb{R}^2 = \text{int}\left(R^2\right) = \text{int}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \emptyset,$$

   which is a contradiction. Hence $\mathbb{R}^2$ cannot be written as a countable union of closed sets each of which has empty interior, as desired.

2. Show that $\mathbb{R}^2$ can be written as the union of countably many sets, each of which has empty interior. We will do this in two ways, first we will set an infinite union and then a countable union.

   First, we consider an infinite union. Define $A_i^c = \{(\pi \cdot i + x, \pi \cdot i + y) : x, y \in \mathbb{Q}\}$ for $i \in \mathbb{N}$. For all $i \in \mathbb{N}$ we have that A_i^c is dense in $\mathbb{R}^2$ because it is just a translation of $\mathbb{Q} \times \mathbb{Q}$, which is dense in $\mathbb{R}^2$. Hence A_i has empty interior for every $i \in \mathbb{N}$. Moreover, $\mathbb{R}^2 = \bigcup_{i=1}^{\infty} A_i$ since $A_i^c \subset A_{2i}$ for every $i \in \mathbb{N} \setminus \{0\}$. Thus we wrote $\mathbb{R}^2$ as the union of countably many sets, each having empty interior.

   We now provide an alternative description, which is much easier if we allow just a finite number of unions. Denote by $\mathbb{I}$ the irrational numbers, notice that all four subsets $\mathbb{Q} \times \mathbb{Q}$, $\mathbb{Q} \times \mathbb{I}$, $\mathbb{I} \times \mathbb{Q}$ and $\mathbb{I} \times \mathbb{I}$ have empty interior, since their complements contain $\mathbb{I} \times \mathbb{I}$, $\mathbb{I} \times \mathbb{Q}$, $\mathbb{Q} \times \mathbb{I}$ and $\mathbb{Q} \times \mathbb{Q}$ respectively, all of which are dense, meaning that the whole complements are also dense. Now, we can write $\mathbb{R}^2 = \mathbb{Q} \times \mathbb{Q} \cup \mathbb{Q} \times \mathbb{I} \cup \mathbb{I} \times \mathbb{Q} \cup \mathbb{I} \times \mathbb{I}$ since the first component of a point $(x, y) \in \mathbb{R}^2$ must be either rational or irrational, and similarly for the second component, and all four cases are included in the union above. Thus we wrote $\mathbb{R}^2$ as the union of finitely many sets, each having empty interior.

Exercise 12.2

Show that if every point of a topological space X has an open neighborhood that is a Baire space, then X itself is a Baire space.

For every $x \in X$ there is, by hypothesis, an open neighborhood U_x that is a Baire space: if we have B_i for $i \in \mathbb{N}$ open and dense in U_x, then $\bigcap_{i\in\mathbb{N}} B_i$ is dense in U_x. Consider now $\{A_i\}_{i\in\mathbb{N}}$ a family of open dense subsets in X. Clearly $A_i \cap U_x$ is open (in the subspace topology) and dense in U_x because it is dense in X. Hence $\bigcap_{i\in\mathbb{N}} A_i \cap U_x = U_x \cap \left(\bigcap_{i\in\mathbb{N}} A_i\right)$ is dense in U_x because it is a Baire space.

Consider now U a nonempty open of X. Because of this, we have that $U \cap U_x \neq \emptyset$ for some $x \in X$. Since $U_x \cap \left(\bigcap_{i\in\mathbb{N}} A_i\right)$ is dense in U_x, we have that $U \cap U_x \cap U_x \cap \left(\bigcap_{i\in\mathbb{N}} A_i\right) \neq \emptyset$, and in particular $U \cap \left(\bigcap_{i\in\mathbb{N}} A_i\right) \neq \emptyset$, meaning that $\bigcap_{i\in\mathbb{N}} A_i$ intersects every nonempty open of X, hence it is dense. This proves that X is a Baire space, as desired.