Topology I - Homework 9 (Final Exam)

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Exercise 13.2

Let $\mathbb{R}^2$ have the standard topology, define $Y = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$ and the surjective function $f : \mathbb{R}^2 \longrightarrow Y$ defined as:

$$f(x,y) = \begin{cases} (x,0) \text{ if } x \neq 0, \\ (0,y) \text{ if } x = 0. \end{cases}$$

We want to show that under the quotient topology induced by f on Y, this is not a Hausdorff space.

First, we give a notion of what the function f does. For every "vertical" line of $\mathbb{R}^2$, that is, a subset of the form $\{a\} \times \mathbb{R}$ with $a \in \mathbb{R} \setminus \{0\}$, the image by f is the point $(a,0)$. When $a = 0$, the function f behaves like a bijection from $\{0\} \times \mathbb{R}$ onto $\{0\} \times \mathbb{R}$.

We now want to see how are the opens in Y around points $(0,b)$ with $b \in \mathbb{R}$. Let $(0,b) \in U$, notice that U is open when $f^{-1}(U)$ is open in $\mathbb{R}^2$. Since $f^{-1}(0,b) = (0,b)$ and $f^{-1}(\{(0,y) \in \mathbb{R}^2 : b - \delta < y < b + \delta\}) = \{(0,y) \in \mathbb{R}^2 : b - \delta < y < b + \delta\}$ for $\delta > 0$, by the discussion above, we need to add points of $\mathbb{R} \times \{0\} \subset Y$ to U to make it open. Note that the open squares trivially generate the same topology than the open circles, hence we may use the former by commodity. Now any basic open square around $(0,b)$ is of the form $S_{0,b}(\epsilon,\delta) = \{(x,y) \in \mathbb{R}^2 : -\epsilon < x < \epsilon, b - \delta < y < b + \delta\}$ with $\epsilon, \delta > 0$, having image:

$$f(S_{0,b}(\epsilon,\delta)) = \{(x,0) \in \mathbb{R}^2 : -\epsilon < x < \epsilon, x \neq 0\} \cup \{(0,y) \in \mathbb{R}^2 : b - \delta < y < b + \delta\},$$

where we define $U_b(\epsilon,\delta) = f(S_{0,b}(\epsilon,\delta))$. However, these basic open squares are not saturated since:

$$f^{-1}(U_b(\epsilon,\delta)) = \{(x,y) \in \mathbb{R}^2 : -\epsilon < x < \epsilon, x \neq 0\} \cup \{(0,y) \in \mathbb{R}^2 : b - \delta < y < b + \delta\}.$$

Nevertheless, we have that $f^{-1}(U_b(\epsilon,\delta))$ is open, hence $U_b(\epsilon,\delta)$ is open, and we can make ϵ and δ as smaller as we desire. Since we are taking preimages of images of basic opens (the squares $S_{0,b}(\epsilon,\delta)$), we obtain that we are adding just enough points of $\mathbb{R} \times \{0\}$ to $\{(0,y) \in \mathbb{R}^2 : b - \delta < y < b + \delta\}$ so that the result, namely $U_b(\epsilon,\delta)$, is open.

The exposition above shows that any open U containing a point $(0,b) \in Y$ must at least contain an open $U_b(\epsilon,\delta)$ for some $\epsilon, \delta > 0$. This means that when we have two distinct points $(0,b_1), (0,b_2) \in Y$ for $b_1, b_2 \in \mathbb{R}$ distinct, any two opens U_1, U_2 containing them respectively, each of them must respectively contain an open of the form $U_{b_1}(\epsilon_1,\delta_1)$, $U_{b_2}(\epsilon_2,\delta_2)$ for certain $\epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0$. This means that one of the epsilons is bigger than the other, without loss of generality we may assume $\epsilon_1 < \epsilon_2$. This means that:

$$\emptyset \neq \{(x,0) \in \mathbb{R}^2 : -\epsilon_1 < x < \epsilon_1, x \neq 0\} \subset U_{b_1}(\epsilon_1,\delta_1) \cap U_{b_2}(\epsilon_2,\delta_2) \subset U_1 \cap U_2,$$

and thus any two such opens intersect in a non empty way. This proves that Y is not Hausdorff, as desired.

Exercise 13.5

Consider the space $\mathbb{R}/\sim$ where two points $x, y \in \mathbb{R}$ are related, $x \sim y$ if and only if $x - y \in \mathbb{Q}$. We want to show that $\mathbb{R}/\sim$ is an uncountable space with trivial topology.

First, suppose $\mathbb{R}/\sim$ was countable. Then we have that $|\mathbb{R}| \leq |R/\sim| \cdot |\mathbb{Q}|$. Since $\mathbb{Q}$ is countable and we have that multiplication of countable infinities is countable, $\mathbb{R}$ would have to be countable. This is a contradiction because we know $\mathbb{R}$ to be uncountable, hence $\mathbb{R}/\sim$ must also be uncountable.

Secondly, consider the quotient map $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\sim$ given by $\pi(x) = [x]$ the class of x inside $\mathbb{R}/\sim$. Since we know that the topology on $\mathbb{R}/\sim$ has as opens the images $\pi(U)$ where U is a saturated open of $\mathbb{R}$, we will show that $U = \mathbb{R}$ is the only non empty saturated opens that there exists. First, since $U \subset \mathbb{R}$ is a nonempty open, it must contain an interval of the form (a, b). Without loss of generality (changing signs if necessary or cropping on the positive part), we may assume that $a, b \geq 0$. Since U is saturated, there must exist $B \subset R/\sim$ with $U = \pi^{-1}(B)$. Since we have an interval in U, for every $x \in (a, b)$ we have $[x] = \pi(x) \in B$, hence:

$$\pi^{-1}(B) \supset \{y \in \mathbb{R} : y = x + q, x \in (a, b), q \in \mathbb{Q}\}.$$

With the above, we now prove $U = \mathbb{R}$. Let $t \in \mathbb{R}$, there are three cases:

1. If $t \in (a, b)$ then clearly $t \in \pi^{-1}(B)$.
2. If $t \geq b$, then there is a $q \in \mathbb{Q}$ with $t - b < q < t - a$ because $\mathbb{Q}$ is dense. This means that $a < t - q < b$ hence $t - q \in (a, b)$ thus $t \in \pi^{-1}(B)$.
3. If $t \leq a$, then there is a $q \in \mathbb{Q}$ with $a - t < q < b - t$ because $\mathbb{Q}$ is dense. This means that $a < t + q < b$ hence $t + q \in (a, b)$ thus $t \in \pi^{-1}(B)$.

Thus in every case, $t \in \pi^{-1}(B)$, meaning that $\mathbb{R} = \pi^{-1}(B) = U$, as desired.

Hence, the only possibilities for opens in $R/\sim$ are $\emptyset$ and $R/\sim$, that is, $R/\sim$ has the trivial topology.

Exercise 14.1

Show that if Y is a Hausdorff space, then $\mathcal{C}(X, Y)$ with the compact-open topology is also a Hausdorff space.

Let $f, g \in \mathcal{C}(X, Y)$ be two distinct continuous maps, that is, there is a point $x \in X$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, there are non empty opens $U, V \subset Y$ with $f(x) \in U$, $g(x) \in V$ and $U \cap V = \emptyset$. Note that in every topological space X, the singletons are always compact (since we can just pick one non empty open as a finite subcover). Now we have that $F_{\{x\},U}$ and $F_{\{x\},V}$ are non empty opens of $\mathcal{C}(X, Y)$ in the compact-open topology, $f \in F_{\{x\},U}$ and $g \in F_{\{x\},V}$. Moreover if $h \in F_{\{x\},U} \cap F_{\{x\},V}$, then $h(x) \in U$ and $h(x) \in V$, but since $U \cap V = \emptyset$, such a continuous function h cannot exist. This means that $F_{\{x\},U} \cap F_{\{x\},V} = \emptyset$ and hence Y is indeed Hausdorff, as desired.

Exercise 15.1

Let X be a topological space and $\{A_j\}_{j\in J}$ a locally finite family of sets in X. Show that $\{\overline{A}_j\}_{j\in J}$ a locally finite family of sets in X.

We know that for every $x \in X$ there exist an open neighborhood U_x of x such that $U_x \cap A_j = \emptyset$ for all $j \in J$ except for $j = i_1, \cdots, i_{n_x}$, where $n_x \in \mathbb{N}$ (for these, we have non empty intersection). This means that $A_j \subset X \setminus U_x$ for $j \neq i_1, \cdots, i_{n_x}$, and since the complement of opens is closed, and $\overline{A}_j$ is the smallest closed containing A_j, we must have that $\overline{A}_j \subset X \setminus U_x$ for $j \neq i_1, \cdots, i_{n_x}$. Moreover, having $A_j \cap U_x \neq \emptyset$ for $j = i_1, \cdots, i_{n_x}$ means $\overline{A}_j \cap U_x \neq \emptyset$ for $j = i_1, \cdots, i_{n_x}$.

We have hence proven that for any $x \in X$, the open neighborhood U_x that intersects non trivially only a finite number n_x of A_j for $j \in J$, namely $j = i_1, \cdots, i_{n_x}$, also intersects non trivially only a finite number n_x of $\overline{A}_j$ for $j \in J$, namely $j = i_1, \cdots, i_{n_x}$. That is, for a given $x \in X$ the same open and number and indexes that make $\{A_j\}_{j\in J}$ a locally finite family of sets are enough to make $\{\overline{A}_j\}_{j\in J}$ a locally finite family of sets, as desired.