Topology II - Homework 1

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Exercise 1

Prove that every faithful and transitive action of an abelian group is a free action. That is, we have an abelian group G acting on a set X which is transitive (for any $x, y \in X$ there is a $g \in G$ with $y = gx$) and faithful (the function $\rho : G \longrightarrow \mathrm{Sym}(X)$ where $\rho(g)$ is the symmetry induced by $g \in G$ on X, is injective). Given $x \in X$, we want to see that $G_x = \{g \in G : gx = x\}$ is trivial.

Clearly we have that the identity element $1 \in G$ belongs to G_x since $1x = x$. Suppose that $g \in G_x$, that is, $gx = x$. By transitivity, for every $y \in X$ there is an element $h_y \in G$ with $h_y y = x$. This means that:

$$gx = x \iff g(h_y y) = h_y y \iff h_y g y = h_y y \iff h_y^{-1} h_y g y = h_y^{-1} h_y y \iff gy = y$$

where we have used that G is abelian and the axioms of an action. This means that $g \in G_y$ for very $y \in X$. Thus:

$$\begin{array}{rcrcl} \rho(1) & : & X & \longrightarrow & X \\ & & y & \longmapsto & y \end{array}, \quad \begin{array}{rcrcl} \rho(g) & : & X & \longrightarrow & X \\ & & y & \longmapsto & gy = y \end{array}$$

and $\rho(1) = \rho(g)$. Since the action is faithful, ρ is injective and $1 = g$. This proves that $G_x = \{1\}$, it is trivial, as desired.

Exercise 2

Let G act by isometries on a proper metric space X. Show that the action is properly discontinuous if and only if, for every point $x \in X$ and $D \in \mathbb{R}^+$ the set $\{g \in G : d(x, gx) < D\}$ is finite.

$\Rightarrow$) Let G acting on X be properly discontinuous, let $x \in X$ and $D \in \mathbb{R}^+$. Notice how if we have $g \in G$ with $d(x, gx) < D$, then $gx \in \overline{B(x, D)}$. Defining $K = \overline{B(x, D)}$, we have that $gx \in K$ by the above and since $x \in K$, we also have $gx \in gK$, hence $gx \in gK \cap K \neq \emptyset$. Thus:

$$\{g \in G : d(x, gx) < D\} \subset \{g \in G : gK \cap K \neq \emptyset\}.$$

Now K is compact because X is proper, and the action being properly discontinuous means that we have that $\{g \in G : gK \cap K \neq \emptyset\}$ is finite, meaning that $\{g \in G : d(x, gx) < D\}$ is finite, as desired.

$\Leftarrow$) Let K be compact and $g \in G$ with $gK \cap K \neq \emptyset$, by this compactness we have that $K = \cup_{i=1}^n B(x_i, d_i)$ for $x_i \in X$, $d_i \in \mathbb{R}^+$ for $i = 1, \ldots, n$ (we can take $d_i = d$ for $i = 1, \ldots, n$ if we wish so, but we do not have to). Now:

$$gK = \cup_{i=1}^n gB(x_i, d_i) = \cup_{i=1}^n B(gx_i, d_i)$$

where we have used in the last equality that G acts by isometries, hence preserves distances: a ball around $z \in X$ is mapped to a ball around gz, with the same radius. Since $gK \cap K \neq \emptyset$, there are elements $x, y \in X$ with $x = gy$, meaning that $d(x, gx) = d(gy, gx) = g(y, x) < \sum_{i=1}^n d_i$, where we have used that G acts by isometries and two elements in K cannot be further away than the sum of the radius of the balls covering K. Hence:

$$\{g \in G : gK \cap K \neq \emptyset\} \subset \left\{ g \in G : d(x, gx) < \sum_{i=1}^n d_i \right\},$$

with $\sum_{i=1}^n d_i \in \mathbb{R}^+$. By hypothesis, the right hand set is finite, hence $\{g \in G : gK \cap K \neq \emptyset\}$ is finite. Since we have proven this for a generic compact set K, we obtain that the action of G on X is properly discontinuous, as desired.