Topology II - Homework 2

Pablo Sánchez Ocal

February 20th, 2017

Exercise 3

Let a group G act (by homeomorphisms) properly discontinuously on X a locally compact, Hausdorff space. We want to show that G/X is Hausdorff.

For this, we first make a few remarks and observations.

1. Since X is locally compact and Hausdorff, we have in virtue of [1, Proposition 11.4 (p. 41)] that for every point $x \in X$ the compact neighborhoods of x form a neighborhood basis at x.

2. An action being properly discontinuous is equivalent to $\{g \in G : gK_1 \cap K_2 \neq \emptyset\}$ being finite for any pair of compact subsets $K_1, K_2 \subset X$:

   $\Rightarrow$) Let $K_1, K_2 \subset X$ be two compact subsets. Since finite union of compacts is compact, the set $K = K_1 \cup K_2$ is compact, and:

   $$\{g \in G : gK_1 \cap K_2 \neq \emptyset\} \subset \{g \in G : gK \cap K \neq \emptyset\}$$

   with the second being finite since the action of G on X is properly discontinuous. Hence, the first set is finite, as desired.

   $\Leftarrow$) Let $K \subset X$ be compact. By taking $K_1 = K = K_2$ and applying the hypothesis, we obtain that $\{g \in G : gK \cap K \neq \emptyset\}$ is finite, as desired.

3. Given any two different points $x, y \in X$, we can find disjoint compact neighborhoods T_x, T_y respectively. To see this, notice that since X is Hausdorff, there are disjoint opens U_x, U_y of x, y respectively (in particular they are neighborhoods of x, y respectively), and since the compact neighborhoods at each point form a neighborhood basis at that point, we have that there are compact neighborhoods $T_x \subset U_x$, $T_y \subset U_y$ containing x, y respectively.

Now we proceed with the proof of G/X being Hausdorff. Let $Gx, Gy \in G/X$ with $Gx \neq Gy$, that is, we have representatives $x, y \in X$ with $\pi(x) = Gx$ and $\pi(y) = Gy$. Since $\pi : X \longrightarrow G/X$ is open, we just need to find opens U_x, U_y containing x, y respectively such that $\pi(U_x) \cap \pi(U_y) = \emptyset$, which happens if and only if $gU_x \cap U_y = \emptyset$ for every $g \in G$. Now since X is locally compact and Hausdorff, we can choose K_x, K_y compact neighborhoods of x, y respectively and there are opens $\tilde{U}_x \subset K_x$, $\tilde{U}_y \subset K_y$ containing x, y respectively. Since the action of G on X is properly discontinuous, the set $\{g \in G : gK_x \cap K_y \neq \emptyset\}$ is finite. If this set is empty, for the opens above we have $g\tilde{U}_x \cap \tilde{U}_y = \emptyset$ for every $g \in G$, and choosing $U_x = \tilde{U}_x$, $U_y = \tilde{U}_y$ we have what we desired.

If we have $\{g \in G : gK_x \cap K_y \neq \emptyset\} = \{g_1, \dots, g_n\}$, notice how we must have $g_i x \neq y$ for every $i = 1, \dots, n$ because $Gx \neq Gy$. Hence we can find for each $i = 1, \dots, n$ disjoint compact neighborhoods T_i^y (with open V_i^x) of y and T_i^x (with open V_i^y) of $g_i x$. Then take $T_x = K_x \cap \bigcap_{i=1}^n g_i^{-1} T_i^x$ and $T_y = K_y \cap \bigcap_{i=1}^n T_i^y$, notice how these are non empty because they contain x and y respectively. Moreover, since G acts by homeomorphisms, we have that the action preserves compactness, hence $\bigcap_{i=1}^n g_i^{-1} T_i^x$ is compact as a finite intersection of compacts, thus both T_x and T_y are compact. They also are neighborhoods, since $U_x = \tilde{U}_x \cap \bigcap_{i=1}^n g_i^{-1} V_i^x \subset T_x$ and $U_y = \tilde{U}_y \cap \bigcap_{i=1}^n V_i^y \subset T_y$ are non-empty opens because they contain x, y respectively and they are finite intersection of opens, always using that the action is by homeomorphisms and hence preserves opens. Finally, by construction, we have that $gU_x \cap U_y = \emptyset$ for every $g \in G$, what we desired.

Exercise 4

Let $f : X \longrightarrow Y$ be a map between topological spaces. Then f is a homotopy equivalence if and only if there exist two maps $g : Y \longrightarrow X$ and $h : Y \longrightarrow Y$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ h \simeq \mathrm{id}_Y$.

$\Rightarrow$) Suppose f is a homotopy equivalence. This means that there is a function $\tilde{f} : Y \longrightarrow X$ such that $\tilde{f} \circ f \simeq \mathrm{id}_X$ and $f \circ \tilde{f} \simeq \mathrm{id}_Y$. Hence choosing $g = \tilde{f} = h$ we obtain the desired maps.

$\Leftarrow$) Suppose we have g and h as above. Notice how we can use the properties of homotopy so that:

$$g = g \circ \mathrm{id}_Y \simeq g \circ f \circ h \simeq \mathrm{id}_X \circ h = h,$$

hence $g \simeq h$ and thus taking $\tilde{f} = g$ we have that $\tilde{f} \circ f = g \circ f \simeq \mathrm{id}_X$ and $f \circ \tilde{f} \simeq f \circ h \simeq \mathrm{id}_Y$. This means that f is a homotopy equivalence and $\tilde{f}$ is a homotopy inverse.

References

[1] Z. Šunić, *Topology - Nuts only*, Class notes of Topology I, Fall 2016.