Topology II - Homework 3

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Exercise 5

1. Let $D \subset S^2$ (the unit sphere in $\mathbb{R}^3$) be a subspace homeomorphic to D^2. Let $C = \{tx : x \in D, t \in \mathbb{R}^+\}$. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ a map such that $f(C \setminus \{0\}) \subset C \setminus \{0\}$. We want to show that there is a non zero $x \in C$ such that 0, x and $f(x)$ are collinear.

   Consider first $\partial C = \{tx : x \in \partial D, t \in \mathbb{R}^+\}$ the boundary of C. Now, note that $\partial C \setminus \{0\}$ is ∂D with a copy of $\mathbb{R}^+$ attached to every point (where given $y \in D$, the way in which $\mathbb{R}^+$ is attached to y is by the straight line going from 0 to y). Since $D \cong S^1$ we have that $\partial C \setminus \{0\}$ is homotopic to S^1 with a copy of $\mathbb{R}^+$ attached to every point as described above. Since this is $\mathbb{R}^2 \setminus \{0\}$, which is homotopic to S^1, we have that $\partial C \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \simeq S^1$.

   If we do not like this explanation, we may consider the maps:

$$\begin{array}{rccc} \alpha : & \partial C \setminus \{0\} & \longrightarrow & \partial D \\ & y = t_y x & \longmapsto & x \end{array}, \qquad \begin{array}{rccc} \beta : & \partial D & \longrightarrow & \partial C \setminus \{0\} \\ & x & \longmapsto & x \end{array},$$

   that are well defined since every $y = \partial C$ can be written uniquely as $y = t_y x$ for certain $t_y \in \mathbb{R}^+$ and $x \in D$ (the uniqueness follows because the intersection of a line passing through the origin of $\mathbb{R}^3$ and S^2 is a single point), and clearly $\partial D \subset \partial C \setminus \{0\}$. These maps are obviously continuous since α is a projection and β is an injection. Now $\alpha \circ \beta = \mathrm{id}_{\partial D}$ and $\beta \circ \alpha \simeq \mathrm{id}_{\partial C \setminus \{0\}}$ via the homotopy:

$$\begin{array}{rccc} H_{\beta\alpha} : & \partial C \setminus \{0\} \times I & \longrightarrow & \partial C \setminus \{0\} \\ & y & \longmapsto & (1-t)\alpha \circ \beta(y) + ty \end{array}$$

   which is continuous as composition of continuous functions. Moreover, $H(y, 0) = \alpha \circ \beta(y)$ and $H(y, 1) = y = \mathrm{id}_{\partial C \setminus \{0\}}(y)$. Hence $\partial C \setminus \{0\} \simeq \partial D \simeq \partial D^2 \simeq S^1$.

   In addition, given any D and f as in the statement of the problem, we know that by a rotation of $\mathbb{R}^3$ and re-sizing, both continuous actions, we have that D is isomorphic to a subspace $\tilde{D} \subset S^2$ entirely contained in the open north hemisphere (in particular $\tilde{C}$ lies completely in the open north hemisphere, and notice how both $\tilde{D}$ and $\tilde{C}$ are closed). By means of the above rotation and re-sizing, we obtain the analogous function $\tilde{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that retains the property of $\tilde{f}(\tilde{C} \setminus \{0\}) \subset \tilde{C} \setminus \{0\}$. Thus we may assume without loss of generality that D and C are entirely contained in the open north hemisphere.

   Once that we know this, we proceed by contradiction. Suppose that for every non zero $x \in C$ we have that 0, x and $f(x)$ are not collinear. Because of this, we have that given $x \in C$ the line defined by $f(x)$ and x does not pass through zero, and since C lies in the open north hemisphere, such line must intersect with ∂C. Starting at $f(x)$ and going in the direction towards x, we name $g(x)$ the intersection point with ∂C. Clearly $g : C \setminus \{0\} \longrightarrow \partial C \setminus \{0\}$ is well defined by the non collinearity of 0, x and $f(x)$, and it is continuous by the continuity of $f(x)$

and the multiplication of a scalar. If $x \in \partial C \setminus \{0\}$, we have that $g(x) = x$ since $x \in \partial C$. Consider now:

$$\begin{array}{rcll} H & : & C \setminus \{0\} \times I & \longrightarrow & C \setminus \{0\} \\ & & (x,t) & \longmapsto & (1-t)x + tg(x) \end{array}$$

which is well defined since $(1-t)x + tg(x)$ parametrizes a segment from x to $\partial C \setminus \{0\}$, that lies inside $C \setminus \{0\}$ because $C \setminus \{0\}$ is convex (we again use that C lies entirely in the north hemisphere). We have that H is continuous by composition of continuous functions and $H(x,0) = x$, $H(x,1) = g(x)$ and for $y \in \partial C \setminus \{0\}$ we have $H(y,1) = g(y) = y$. Hence H defines a homotopy between $\mathrm{id}_{C \setminus \{0\}}$ and g meaning that $\partial C \setminus \{0\}$ is a strong deformation retract of $C \setminus \{0\}$.

This means that $C \setminus \{0\}$ and $\partial C \setminus \{0\}$ have the same fundamental group. However, this is a contradiction since $C \setminus \{0\}$ retracts to D in the obvious way given by its definition, and D is isomorphic to D^2 hence contractible, meaning that $\pi_1(C \setminus \{0\}) = \{0\}$, and we already saw that $\partial C \setminus \{0\}$ is homotopic to S^1, hence $\pi_1(\partial C \setminus \{0\}) = \mathbb{Z}$, and they are clearly different. Hence we must have that there is at least one $x_0 \in C \setminus \{0\}$ such that 0, x_0, x_0 and $f(x_0)$ are collinear, the desired result.

2. We want to show using the above result that if $A \in M_3(\mathbb{R})$ with positive entries, then it has at least one eigenvector with all entries real and positive whose eigenvalue is also real and positive.

   We define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as $f(v) = Av$ for any given $v \in \mathbb{R}^3$. We will use $D = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, that is, the closed north hemisphere, which is clearly isomorphic to D^2, meaning that $C = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$. Since A has positive entries and any $v \in C \setminus \{0\}$ has at least one positive entry, we have that $f(v) = Av$ has at least one positive entry, meaning that $f(v) \in C \setminus \{0\}$. Thus f satisfies the hypothesis in the section above, meaning that there is a non-zero point $v_0 \in C$ with 0, v_0, $f(v_0)$ colinear. This means that the line defined by v_0 and $f(v_0)$ goes through the origin, thus they are proportional to each other and since both have real positive entries, the proportionality factor must be positive, thus there exists $\lambda \in \mathbb{R}^+$ positive such that $f(v_0) = \lambda v_0$. This means that $\lambda v_0 = f(v_0) = Av_0$ and thus v_0 and λ are the respective eigenvector (with all real positive entries) and eigenvalue (real and positive) that we desired. Notice in particular that since $v_0 \neq 0$ we have $f(v_0) \neq$ thus $\lambda > 0$.

Exercise 6

1. Let $f : D^2 \longrightarrow D^2$ be a map such that $f(x) = x$ for $x \in S^1$. Then there exists $z \in D^2 \setminus S^1$ such that $f(z) = z$. We claim that this is false, and we will build a counterexample.

   First, we note that this is already not true in lower dimensions, since the continuous function $g : I \longrightarrow I$ given by $g(t) = t^2$ for $t \in I$ is such that $g(0) = 0$, $g(1) = 1$ but $g(t) \neq t$ for every $t \in (0, 1)$. What we want is to generalize this idea by building a function that applies this map in each vertical sections of the disc.

   First, we parametrize the vertical sections of the disc as segments. Let $(x, y) \in D^2$, fix $x \in [-1, 1]$, we want to parametrize the vertical segment via $y_x(t)$. Since we know that $x^2 + y_x(t)^2 \leq 1$ and at the top boundary and bottom boundary of the disc (which we want to be the origin and end points of our segment respectively) we have $x^2 + y_x(t)^2 = 1$, we must have that at the endpoints $y_x(t) = \pm\sqrt{1 - x^2}$ for certain $0 \leq t \leq 1$. By setting $y_x(0) = -\sqrt{1 - x^2}$ and $y_x(1) = \sqrt{1 - x^2}$, we parametrize the vertical segment inside the disc going from $(x, -\sqrt{1 - x^2})$ to $(x, \sqrt{1 - x^2})$ as:

   $$y_x(t) = -(1 - t)\sqrt{1 - x^2} + t\sqrt{1 - x^2} \quad \text{with} \quad 0 \leq t \leq 1.$$

   We immediately check that $y_{\pm 1}(t) = 0$ for every $0 \leq t \leq 1$, that is, when $x = \pm 1$ we indeed are at the points $(-1, 0)$ and $(1, 0)$, and our parametrization has shrunk the segment to a point. We clearly have that $D^2 = \{(x, y_x(t)) : -1 \leq x \leq 1, 0 \leq t \leq 1\}$. Moreover, by construction and the uniqueness of the parametrization of $y_x(t)$ for both $t \in [0, 1]$ and $x \in (-1, 1)$, we have that given $(x, y) \in D^2 \setminus \{(1, 0), (0, 1)\}$ there is only one value of $t \in [0, 1]$ such that $y = y_x(t)$ and thus $(x, y) = (x, y_x(t))$. Notice that we have seen above that if $x = \pm 1$ then we do not have uniqueness for $t \in [0, 1]$. However, if $(x, y) \in D^2 \setminus S^1$, we do have this uniqueness in both $t \in (0, 1)$ and $x \in (-1, 1)$.

   Consider the function:

   $$\begin{array}{cccc} f & : & D^2 & \longrightarrow & D^2 \\ & & (x, y_x(t)) & \longrightarrow & (x, y_x(t^2)) \end{array}$$

   which is continuous as componentwise composition of continuous functions. Since by the above there is a bijection between $(x, t) \in (0, 1) \times (-1, 1)$ and $(x, y) \in D^2 \setminus S^1$, if $(x, y_x(t)) \in D^2 \setminus S^2$ then $(x, y_x(t)) \neq (x, y_x(t^2))$ because $y_x(t)) \neq y_x(t^2)$ because $t^2 \neq t$ because $t \in (0, 1)$ and $x \in (-1, 1)$. Thus f has no fixed points in $D^2 \setminus S^1$. Moreover, suppose $(x, y) \in S^1$. This means by construction of our parametrization that either $x = \pm 1$ and we already saw that $f(\pm 1, y_{\pm}(t)) = (\pm 1, y_{\pm}(t^2)) = (\pm 1, 0) = (\pm 1, y_{\pm 1}(t))$ for every $t \in [0, 1]$, either $t = 0$ and $f(x, y_x(0)) = (x, y_x(0^2)) = (x, y_x(0))$ for every $x \in [-1, 1]$, or $t = 1$ and $f(x, y_x(1)) = (x, y_x(1^2)) = (x, y_x(1))$ for every $x \in [-1, 1]$. In any of the four cases, we have that $f(x, y) = (x, y)$ for $(x, y) \in S^1$.

Thus $f : D^2 \longrightarrow D^2$ is a map such that $f(x, y) = (x, y)$ for $(x, y) \in S^1$ and $f(z) \neq z$ for every $z \in D^2 \setminus S^1$, thus it is a desired counterexample of the statement.

2. Let $f : D^2 \longrightarrow D^2$ be a map such that $f(x) = x$ for $x \in S^1$, then f is surjective. We claim that this is true.

   We proceed by contradiction. Suppose there is $y \in D^2$ but $y \notin \text{im}(f)$. By composition with a translation if necessary, we may assume that $y = 0$. We define the function:

$$\begin{array}{cccc} g & : & D^2 & \longrightarrow & S^1 \\ & & x & \longrightarrow & f(x)/||f(x)||_2 \end{array}$$

   which is well defined since $0 \notin \text{im}(f)$ thus $f(x) \neq 0$ for every $x \in D^2$, and it is clearly continuous as composition of continuous functions. Consider now:

$$\begin{array}{cccc} H & : & D^2 \times I & \longrightarrow & D^2 \\ & & (x, t) & \longmapsto & (1-t)x + tg(x) \end{array}$$

   which is well defined since $(1-t)x+tg(x)$ parametrizes a segment between elements in D^2, that lies inside D^2 because it is convex. We have that H is continuous by composition of continuous functions and $H(x, 0) = x$, $H(x, 1) = g(x)$ and for $y \in S^1$ we have $H(y, 1) = g(y) = f(y)/||y||_2 = y/1 = y$. Hence H defines a homotopy between id_{D^2} and g meaning that S^1 is a strong deformation retract of D^2.

   This is a contradiction since it would mean that D^2 and S^1 have the same fundamental group. However, we know that D^2 is contractible, meaning that $\pi_1(D^2) = \{0\}$, and $\pi_1(S^1) = \mathbb{Z}$, where they are clearly different. Hence we must have that every $y \in D^2$ is $y \in \text{im}(f)$, thus f is surjective, as desired.