Topology II - Homework 4

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Exercise 7

Let F be a topological space, $X \subset F$. We say that F is free topological space over X if, for every topological space Y and every (set) function $f : X \longrightarrow Y$ can be uniquely extended to a continuous function $\tilde{f} : F \longrightarrow Y$. Describe precisely the free topological spaces.

Consider the topological space $X = F$ with the trivial topology. Clearly any set function $f : X \longrightarrow Y$ to a given topological space Y can be extended to a continuous function $\tilde{f} : F \longrightarrow Y$ with $\tilde{f}|_F = f$, namely $\tilde{f} = f$; the discrete topology on X assures that the preimage of any subset of Y is open in X, hence the preimage of any open is also open and $\tilde{f} = f$ is continuous. Moreover, since restricted to X we need that $\tilde{f}$ behaves like f, but the domain of f is the whole F, this extension is unique; given any point $x \in F$ we have $x \in X$ and thus $\tilde{f}(x) = f(x)$, thus $\tilde{f}$ is uniquely defined. Finally, since the property above is a so called universal property, we know by categorical theoretical reasons that the object defined by it (in this case the free topological space) is unique up to unique homeomorphism (note that this can also readily be proven by simply assuming that we have two different objects that satisfy the property above and apply it successively from one to the other. However, in this particular case, the proof gets quite messy). This means that the pair F with the discrete topology is a precise description of what we wanted.

Exercise 8

Let:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

show that $G = \langle A, B \rangle$ is free over $\{A, B\}$ by considering the usual linear action of $\mathrm{SL}_2(\mathbb{Z})$ on the real plane $\mathbb{R}^2$ and finding a ping-pong table for G within the real plane $\mathbb{R}^2$.

First, we notice that we have for any $n \in \mathbb{Z} \setminus \{0\}$:

$$A^n = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}, \quad B^n = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}.$$

We consider $X_A = \{(x,y) \in \mathbb{R}^2 : |x| > |y|\}$, $X_B = \{(x,y) \in \mathbb{R}^2 : |x| < |y|\}$. Otherwise said, we divide $\mathbb{R}^2$ by the lines $y = x$, $y = -x$ and consider X_A the points in the area closer to the x axis, X_B the points in the area closer to the y axis (both axis and the line themselves are not included anywhere). This is our ping-pong table:

1. $X_A \cap X_B = \emptyset$ since given two distinct positive numbers x and y, we have that one must be greater than the other in absolute value, so either $|x| > |y|$ and we are in X_A or $|x| < |y|$ and we are in X_B.

2. Let $(x,y) \in X_B$, that is, $|x| < |y|$. We have that:

$$A^n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2ny \\ y \end{bmatrix},$$

   that has $|x + 2y| > |y|$ regardless of the signs of x and y, since adding y to x at least twice with any sign already overpowers the single y in absolute value of the second component, thus belongs in X_A, and $A^n(X_B) \subset X_A$ for every $n \in \mathbb{Z}$.

3. Let $(x,y) \in X_A$, that is, $|x| > |y|$. We have that:

$$B^n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y + 2nx \end{bmatrix},$$

   that has $|x| < |y + 2nx|$ regardless of the signs of x and y by the same reason as above: adding x to y at least twice with any sign already overpowers the single x in absolute value of the first component, thus belongs in X_B, and $B^n(X_A) \subset X_B$ for every $n \in \mathbb{Z}$.

Thus we are under the hypothesis of the Ping-Pong Lemma, that guarantees that G is free on $\{A, B\}$, as desired.