Topology II - Final Homework

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May 14th, 2017

Exercise 1

1. Provide a presentation of the fundamental group G of the figure 8 knot. Using the diagram in Figure 1 and the method explained in class, we obtain that we have a, b, c, d as generators and $ac = cd$, $ad = ca$, $db = cd$, $ab = bd$, thus:

$$G = \pi_1(K) = \langle a, b, c, d | ac = cd, ad = ca, db = cd, ab = bd \rangle.$$

First, we notice that the last equality is a consequence of the other three: we can write $b = c^{-1}ac$ and $d = a^{-1}ca$ and $b = d^{-1}cd$ so that $ab = ad^{-1}cd = aa^{-1}c^{-1}aca^{-1}ca = c^{-1}aca^{-1}ca = bd$. Hence deleting the fourth equality and using the first two to express b and d in terms of a and c, we obtain:

$$G = \langle a, c | a^{-1}cac^{-1}ac = ca^{-1}ca \rangle = \langle x, y | yxy^{-1}xy = xyx^{-1}yx \rangle.$$

2. We show that K is not the unknot by finding a finite non-abelian quotient of G. For this, we try the dihedral groups of small order $D_n = \langle x, y | x^2 = 1 = y^2, (xy)^n = 1 \rangle$. In particular, identifying $x \in G$ with $x \in D_n$ and $y \in G$ with $y \in D_n$, we obtain that the relation $yxy^{-1}xy = xyx^{-1}yx$ in D_n must be:

$$yxy^{-1}xy \stackrel{?}{=} xyx^{-1}yx \iff yxyxy = xyxyx \iff xyxyxyxyxy = 1 \iff (xy)^5 = 1$$

meaning that our candidate is D_5. Now, consider the so called in class "adding a relator crushes one group" Lemma, stating that for every X, R, S, the identity on X can be extended to a surjective homomorphism $\langle X|R \rangle \longmapsto \langle X|R \cup S \rangle$. Applied to our case (the hypothesis hold by the above reasoning) we obtain $\phi : G \longrightarrow D_5$ a surjective homomorphism, meaning that by the First Isomorphism Theorem $G/\ker(\phi) \cong D_5$, and we found a finite non-abelian quotient of G. Since quotient of abelian groups are abelian, we must have that G is non-abelian. However, the unknot $\mathbb{S}^1$ has an abelian fundamental group, hence $K \ncong \mathrm{S}^1$, as desired.

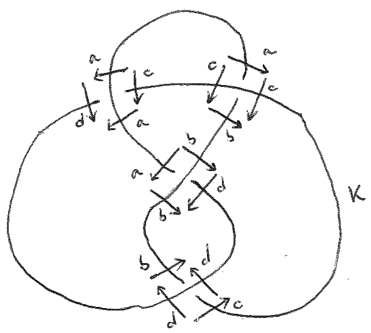

Figure 1: The figure 8 knot with intersections labeled.

Exercise 2

Denote the unit open ball in Rn by Bn, its closure by Dn, and its boundary by Sn-1. Show that, for every pair of points y and z in Bn, there exists a homeomorphism from Dn to Dn that maps y to z and is the identity on the boundary Sn-1

1. Let $\mathbb{B}^n$ be the unit open ball in $\mathbb{R}^n$, $\overline{\mathbb{B}^n} = \mathbb{D}^n$ and $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$. Let $y, z \in \mathbb{B}^n$, we construct a homeomorphism $\varphi : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ with $\varphi|_{\mathbb{S}^{n-1}} = \mathrm{id}|_{\mathbb{S}^{n-1}}$ and $\varphi(y) = z$.

   Given $y, z \in \mathbb{B}^n$, if they are equal, the identitiy works, so we will consider them differently, in particular let L be the line containing them. Take any $x \in \mathbb{D}^n \setminus L$, we parametrize the half-line from y towards x as $L_x(t) = (1-t)y + t\tilde{x}$ with $t \in [0, \infty)$, where $\tilde{x}$ is the unique point where such half line that intersects $\partial\mathbb{B}^n$. We note that the whole line lies inside $\mathbb{D}^n$. Since by definition x belongs to this line, we have that there is a time $t_x(y) \in (0, 1]$ such that $L_x(t_x(y)) = x$. We set:

$$\begin{array}{rccc} \varphi & : & \mathbb{D}^n \setminus L & \longrightarrow & \mathbb{D}^n \setminus L \\ & & x & \longmapsto & (1 - t_x(y))z + t_x(z)\tilde{x} \end{array}.$$

   Since the expression for $L_x(t)$ is continuous in t, y and $\tilde{x}$ whenever outside L, we have that φ is continuous. Moreover, it has inverse:

$$\begin{array}{rccc} \varphi^{-1} & : & \mathbb{D}^n \setminus L & \longrightarrow & \mathbb{D}^n \setminus L \\ & & x & \longmapsto & (1 - t_x(z))y + t_x(z)\tilde{x} \end{array}.$$

   where we have done the same construction buy naming the corresponding time $t_x(z)$. This yields its inverse since doing the same construction starting with z instead of y, by definition $\tilde{x}$ is the unique point in the intersection of the half-line from z to $\varphi(x)$ with $\partial\mathbb{B}^n$, meaning that $t_{\varphi(x)}(z) = t_x(y)$ and thus $\varphi^{-1}(\varphi(x)) = (1 - t_{\varphi(x)(z)})y + t_{\varphi(x)(z)}\tilde{x} = L_x(t_x) = x$ (in particular φ is bijective), and the other equality follows by the same reasoning. If $x \in \mathbb{S}^{n-1} \setminus L$ we have $x = \tilde{x}$ and thus $t_x = 1$ so $\varphi(x) = x$.

   We now consider what happens in L. Notice that in $L \setminus \{y\}$ we have that φ is well defined and continuous and lands in $L \setminus \{z\}$ since we never have time zero, and similarly in $L \setminus \{z\}$ we have that φ^{-1} is well defined and continuous and lands in $L \setminus \{y\}$. Moreover we have the limits $\varphi(x) \to z$ when $x \to y$ and $\varphi^{-1}(x) \to y$ when $x \to z$ in every open (by continuity in every line, including L) thus we can extend $\varphi(y) = z$ and $\varphi^{-1}(z) \to y$ to obtain $\varphi : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ a bijective continuous map with continuous inverse, $\varphi|_{\mathbb{S}^{n-1}} = \mathrm{id}|_{\mathbb{S}^{n-1}}$ and $\varphi(y) = z$, as desired.

2. From now onward, let M be a path-connected n-manifold for $n > 0$ with $\partial M = \emptyset$. We show that for every point $x \in M$ there is an open neighborhood $U_x \ni x$ such that for every pair $y, z \in U_x$ there is a homeomorphism from M to M mapping y to z.

   Since M is a manifold, it is second countable and thus has a countable basis, say $\mathcal{B} = \{U_i\}_{i \in \mathbb{N}}$. Moreover, given $x \in M$, since M is locally euclidean there is

$W_x \ni x$ an open in M homeomorphic to a ball $\mathbb{B}^n(x,\delta)$. Since $\mathcal{B}$ is a basis, we can assume (making δ as small as necessary) that $\overline{W_x} \subset U_j$ for certain $j \in \mathbb{N}$. Setting $\psi : W_x \xrightarrow{\cong} \mathbb{B}^n(x,\delta)$ the homeomorphism above restricted as necessary, we consider $U_x = \psi^{-1}(\mathbb{B}^n(x,\epsilon))$ for certain $0 < \epsilon < \delta$. Given any $y, z \in U_x$ we have $\psi(y), \psi(z) \in \mathbb{B}^n(x,\epsilon)$, thus using the section above we know that there exists a homeomorphism $\varphi : \mathbb{D}^n(x,\epsilon) \xrightarrow{\cong} \mathbb{D}^n(x,\epsilon)$ with $\varphi(\psi(y)) = \psi(z)$. Consider $\phi = \psi^{-1} \circ \varphi|_{\mathbb{B}^n(x,\epsilon)} \circ \psi : U_x \xrightarrow{\cong} U_x$ a homeomorphism with $\phi(y) = z$. This trivially extends to $\phi : \overline{U_x} \xrightarrow{\cong} \overline{U_x}$ and since ϕ is defined through maps that are the identity on the boundary, we have that $\phi|_{\partial U_x} = \mathrm{id}|_{\partial U_x}$. Now we use the Gluing Lemma to extend to $\Phi : \overline{W_x} \xrightarrow{\cong} \overline{W_x}$ by setting $\Phi(u) = u$ if $u \in \overline{W_x} \setminus U_x$ and $\Phi(u) = \phi(u)$ if $u \in \overline{U_x}$: clearly $\overline{W_x} \setminus U_x$ and $\overline{U_x}$ are closed and both are the identity on ∂U_x, their intersection, so Φ is continuous and in fact a homeomorphism (it has continuous inverse the same construction using ϕ^{-1}). We use the Gluing Lemma again to extend it to $\Theta : U_j \xrightarrow{\cong} U_j$ by setting $\Theta(u) = u$ if $u \in U_j \setminus \overline{U_x}$ and $\Theta(u) = \Phi(u)$ if $u \in W_x$: clearly $U_j \setminus \overline{U_x}$ and W_x are open and both are the identity on $W_x \setminus \overline{U_x}$, their intersection, so Θ is continuous and in fact a homeomorphism (it has continuous inverse the same construction using Φ^{-1}). Finally, we use the Gluing Lemma again to extend it to $\Psi : M \xrightarrow{\cong} M$ by setting $\Psi(u) = u$ if $u \in U_i$ with $i \neq j$ and $\Psi(u) = \Theta(u)$ if $u \in U_j$: clearly U_i are open for $i \in \mathbb{N}$, and since we chose U_x so that its closure was fully contained only in U_j, the intersection $U_i \cap U_k$ never contains U_x and everything is the identity there, so Ψ is continuous and in fact a homeomorphism (it has continuous inverse the same construction using Θ^{-1}). By construction, we see that $\Psi(y) = \phi(y) = z$, hence we obtained the desired result.

3. We show that for every pair of points $y, z \in M$ there is a homeomorphism from M to M mapping y to z. Here we use that M is path connected, say $\gamma : [0,1] \longrightarrow M$ being the path connecting y to z with $\gamma(0) = y$ and $\gamma(1) = z$. Since $[0,1]$ is compact we know that $\mathrm{im}(\gamma)$ is compact, and using the notation and the result in the section above we have that clearly $\{U_x\}_{x \in \mathrm{im}(\gamma)}$ is an open cover of $\mathrm{im}(\gamma)$ (here $U_x \ni x$ is such that for every pair $y, z \in U_x$ there is a homeomorphism from M to M mapping y to z, using the section above). Hence there is a finite open subcover $\{U_i\}_{i=1}^n$ of $\mathrm{im}(\gamma)$, where the order is chosen as the opens that we find when $t \in [0,1]$ increases, that is, we have $y \in U_1$, $z \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$. Picking $x_i \in U_i \cap U_{i+1}$ for $i = 1, \ldots, n-1$, we have that $y, x_1 \in U_1$, $x_1, x_2 \in U_2$, ..., $x_{n-1}, z \in U_n$. Applying the section above, there are homeomorphisms $\Phi_1 : M \longrightarrow M$, $\Phi_2 : M \longrightarrow M$, ..., $\Phi_n : M \longrightarrow M$ with $\Phi_1(y) = x_1$, $\Phi_2(x_1) = x_2$, ..., $\Phi_n(x_{n-1}) = z$. Hence the map $\Phi = \Phi_n \circ \cdots \circ \Phi_1 : M \longrightarrow M$ is a homeomorphism with $\Phi(y) = z$, as desired.