Algebra I - Homework 1

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Exercise 1

We have a group G, a set A and G^A the set of all maps $f : A \longrightarrow G$ equipped with the pointwise multiplication: $f, g \in G^A$, then we define $f \cdot g \in G^A$ by setting $f \cdot g(x) = f(x)g(x)$ for every $x \in A$. For it to be a group we just have to check:

1. Associativity: for $f, h, g \in G^A$, we have:

$$(f \cdot g) \cdot h(x) = (f \cdot g(x))h(x) = f(x)g(x)h(x) = f(x)(g \cdot h(x)) = f \cdot (g \cdot h)(x).$$

2. Identity element: consider the map $id \in G^A$ given by $id(x) = e$ for every $x \in A$, now for any $f \in G^A$ we have:

$$f \cdot id(x) = f(x)id(x) = f(x)e = f(x) = ef(x) = id(x)f(x) = id \cdot f(x).$$

3. Inverse: let $f \in G^A$, consider the map $g \in G^A$ given by $g(x) = f(x)^{-1}$ for every $x \in A$, now we have $g = f^{-1}$ since:

$$f \cdot g(x) = f(x)g(x) = f(x)f(x)^{-1} = e = f(x)^{-1}f(x) = g(x)f(x) = g \cdot f(x).$$

Exercise 2

When G is a group with $g^2 = 1$ for every $g \in G$, in particular we have $g = g^{-1}$ for every $g \in G$, and thus for every pair $g, h \in G$ we have $gh \in G$ meaning that:

$$ghgh = (gh)^2 = 1 \Longrightarrow gh = (gh)^{-1} = h^{-1}g^{-1} = hg,$$

making G commutative.

Exercise 3

We have a group G and $A \subset G$ an arbitrary subset. We want to verify that $Z(A) = \{g \in G : gh = hg\, \forall h \in A\}$ is a subgroup of G, so we just have to check:

1. Closed under multiplication: let $g_1, g_2 \in Z(A)$, then $g_1 g_2 \in Z(A)$ since:

$$(g_1 g_2)h = g_1(g_2 h) = g_1(hg_2) = (g_1 h)g_2 = (hg_1)g_2 = h(g_1 g_2) \text{ for every } h \in A.$$

2. Closed under inverses: let $g \in Z(A)$, then $g^{-1} \in Z(A)$ since:

$$gh = hg \Longrightarrow hg^{-1} = g^{-1}(gh)g^{-1} = g^{-1}(hg)g^{-1} = g^{-1}h.$$

Exercise 4

We want the kernel and image of the homomorphism $f : D_6 \longrightarrow S_3$ defined by the permutations on the diagonals (see Figure 1) induced by the symmetries of the hexagon.

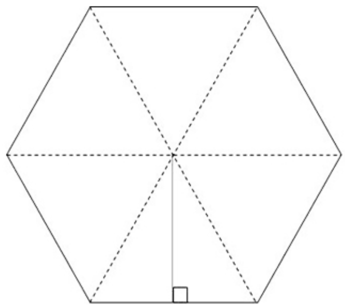

Figure 1: The three diagonals of the regular hexagon, [1].

We begin by making a list of the elements of D_6, where we taking as starting point the right vertex in Figure 1 and measure the angles counterclockwise. We have six rotations r_α with $\alpha \in \{0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3\}$, three mirror symmetries along the edges e_β with $\beta \in \{\pi/6, \pi/2, 5\pi/6\}$ and three mirror symmetries along the diagonals d_γ with $\gamma \in \{0, \pi/3, 2\pi/3\}$. These diagonals will be referred as 1, 2 and 3 respectively when considered as elements of S_3, to ease notation.

We now proceed by computing the images of the elements in D_6 by f (since we are said that f is a homomorphism, that fact will not be verified):

$$\begin{array}{rcccc} f & : & D_6 & \longrightarrow & S_3 \\ & & r_0 & \longrightarrow & \text{id} \\ & & r_{\pi/3} & \longrightarrow & (1,2,3) \\ & & r_{2\pi/3} & \longrightarrow & (1,3,2) \\ & & r_\pi & \longrightarrow & \text{id} \\ & & r_{4\pi/3} & \longrightarrow & (1,2,3) \\ & & r_{5\pi/3} & \longrightarrow & (1,3,2) \\ & & e_{\pi/6} & \longrightarrow & (1,2) \\ & & e_{\pi/3} & \longrightarrow & (2,3) \\ & & e_{5\pi/6} & \longrightarrow & (1,2) \\ & & d_0 & \longrightarrow & (2,3) \\ & & d_{\pi/3} & \longrightarrow & (1,3) \\ & & d_{2\pi/3} & \longrightarrow & (1,2) \end{array}$$

meaning that $\text{im}(f) = S_3$ and $\ker(f) = \{r_0, r_\pi\}$, that is, the homomorphism is surjective and the kernel is the group formed by the identity and the rotation of an angle π (which is the only rotation of order 2).

Exercise 5

The group of quaternions $\mathbb{H}$ cannot be isomorphic to D_4 the group of symmetries of the square: $i \in \mathbb{H}$ is such that $i^2 = -1$, but there is no element in D_4 whose square is the identity element:

1. Any rotation r_α with $\alpha \in \{0, \pi/2, \pi, 3\pi/2\}$ has $r_\alpha^2 = r_{2\alpha}$ (obviously r_0 is the identity, different from $-r_0$, and we are working with α modulo a factor 2π), which are never $-r_0$.

2. Any mirror image along the diagonals or the edges m_β with $\beta \in \{1, 2, 3, 4\}$ has $d_\beta^2 = r_0$.

Exercise 6

We want to give subgroups of $\mathrm{GL}_2(\mathbb{R})$ isomorphic to:.

1. $(\mathbb{R},+)$: consider the set $\mathrm{SUT}_2(\mathbb{R})$ (upper triangular matrices with determinant 1) formed by matrices with 1 in the diagonal and $r \in \mathbb{R}$ in the upper right entry, we have the obvious bijection:

$$\begin{array}{rccc} f & : & (\mathbb{R},+) & \longrightarrow & \mathrm{SUT}_2(\mathbb{R}) \\ & & r & \longmapsto & \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \end{array}$$

and equipping $\mathrm{SUT}_2(\mathbb{R})$ with the usual matrix multiplication, it becomes a group:

   (a) $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r+s \\ 0 & 1 \end{pmatrix}$,

   (b) $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$,

   (c) $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$,

and obviously f becomes an isomorphism, as we wanted.

2. $(\mathbb{R}^*,\cdot)$: consider the set $\mathrm{FE}_2(\mathbb{R})$ formed by matrices with $r \in \mathbb{R}^*$ in the first entry in the diagonal and 1 in the second, we have the obvious bijection:

$$\begin{array}{rccc} f & : & (\mathbb{R}^*,\cdot) & \longrightarrow & \mathrm{FE}_2(\mathbb{R}) \\ & & r & \longmapsto & \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

and equipping $\mathrm{FE}_2(\mathbb{R})$ with the usual matrix multiplication, it becomes a group:

   (a) $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} rs & 0 \\ 0 & 1 \end{pmatrix}$,

   (b) $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$,

   (c) $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/r & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$,

and obviously f becomes an isomorphism, as we wanted.

References

[1] ontrack-media.net, *gm2l7test.html*, cited 2016. [Available online at http://www.ontrack-media.net/geometry/Geometry%20Tests/gm2l7test.html.]