Algebra I - Homework 6

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Exercise 1

Let I be an ideal of a commutative ring R. Show that $J = \{r \in R : r^N \in I$ for some $n \in \mathbb{N}\}$ is an ideal. We notice that given any $r \in R$, for every $a \in J$ (say $a^n \in I$ for some $n \in \mathbb{N}$) we have that (since R is commutative):

$$(ra)^n = r^n a^n \in I$$

since $r^n \in R$ and $a^n \in I$ an ideal. Thus $ra \in J$ and J is an ideal (because R is commutative).

Exercise 2

Let R be a commutative ring, I, J ideals with $I + J = R$. We prove that $IJ = I \cap J$:

$\subseteq$) Let $ab \in IJ$ (that is $a \in I$, $b \in J$). Then clearly $ab \in I$ since $a \in I$, $b \in R$ and $ab \in J$ since $a \in R$, $b \in J$, thus $ab \in I \cap J$. Hence, any finite sum of elements of this form also belongs in $I \cap J$, obtaining $IJ \subset I \cap J$.

$\supseteq$) Let $r \in I \cap J$, that is, $r \in I$, $r \in J$. Since $I + J = R$ with unity, there are elements $a \in I$, $b \in J$ with $a + b = 1$. Multiplying by r, we obtain $ar + rb = r$ with $br, rb \in IJ$ (we used that R is commutative) and thus $r \in IJ$ as desired.

We now provide an example where this does not hold. Let $R = \mathbb{Z}$ commutative, consider $I = 2\mathbb{Z} = J$, we have that $IJ = 4\mathbb{Z} \neq 2\mathbb{Z} = I \cap J$.

Exercise 3

Let $e \in R$ be an idempotent and central element. We prove that Re and $R(1-e)$ are ideals of R:

1. For Re, let $s, r \in R$, we have that:

$$\begin{aligned} (se)r = ser = sre = (sr)e \in Re \\ r(se) = rse = (rs)e \in Re \end{aligned}$$

   because e belongs to the center.

2. For $R(1-e)$, note that $1-e$ belongs to the center since both 1 and e belong to the center. Let $s, r \in R$, we have that:

$$\begin{aligned} (s(1-e))r = s(1-e)r = sr(1-e) = (sr)(1-e) \in R(1-e) \\ r(s(1-e)) = rs(1-e) = (rs)(1-e) \in R(1-e) \end{aligned}$$

   because $1-e$ belongs to the center.

We now prove that $R \cong Re \times R(1-e)$. For this, notice that:

$$(1-e)e = e(1-e) = e - e^2 = 0 \text{ and } (1-e)(1-e) = 1 - e - e - e^2 = 1 - e.$$

Now define the natural map:

$$\begin{array}{cccc} \varphi & : & R & \longrightarrow & Re \times R(1-e) \\ & & r & \longmapsto & (re, r(1-e)) \end{array}$$

which is a ring homomorphism since for $r, s \in R$:

$$\begin{aligned} \varphi(rs) = (rse, rs(1-e)) \\ \varphi(r)\varphi(s) = (rese, r(1-e)s(1-e)) = (rse, rs(1-e)) \end{aligned}$$

where we have used the remarks above, and:

$$\begin{aligned} \varphi(r+s) = ((r+s)e, (r+s)(1-e)) \\ \varphi(r) + \varphi(s) = (re, r(1-e)) + (se, s(1-e)) = ((r+s)e, (r+s)(1-e)). \end{aligned}$$

Moreover, φ is injective since if $r, s \in R$ with $\varphi(r) = \varphi(s)$ then:

$$(re, r(1-e)) = (se, s(1-e)) \iff re = se, r - re = r(1-e) = s(1-e) = s - se \iff r = s$$

and is surjective because given $(re, s(1-e)) \in R \times R(1-e)$ we have $re + s(1-e) \in R$:

$$\varphi(re + s(1-e)) = (ree + s(1-e)e, re(1-e) + s(1-e)(1-e)) = (re, s(1-e)).$$

This means that φ defines an isomorphism $R \cong Re \times R(1-e)$, as desired.

Exercise 4

Consider U the ring of real 2×2 upper triangular matrices. Let:

$$I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

We have that I is an ideal of U, since for any $a, r, s, t \in \mathbb{R}$ we have:

$$\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ra \\ 0 & 0 \end{pmatrix} \in I$$
$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & at \\ 0 & 0 \end{pmatrix} \in I.$$

To see that $U/I \cong \mathbb{R} \times \mathbb{R}$, define the natural map:

$$\begin{array}{rccc} \varphi & : & U & \longrightarrow & \mathbb{R} \times \mathbb{R} \\ & & \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} & \longmapsto & (r, t) \end{array}$$

we have that φ is a ring homomorphism since for every $r_1, r_2, s_1, s_2, t_1, t_2$ we have:

$$\varphi\left(\begin{pmatrix} r_1 & s_1 \\ 0 & t_1 \end{pmatrix} \begin{pmatrix} r_2 & s_2 \\ 0 & t_2 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} r_1 r_2 & r_1 s_2 + s_1 t_2 \\ 0 & t_1 t_2 \end{pmatrix}\right) = (r_1 r_2, t_1 t_2)$$
$$\varphi\left(\begin{pmatrix} r_1 & s_1 \\ 0 & t_1 \end{pmatrix}\right) \varphi\left(\begin{pmatrix} r_2 & s_2 \\ 0 & t_2 \end{pmatrix}\right) = (r_1, t_1)(r_2, t_2) = (r_1 r_2, t_1 t_2)$$

and:

$$\varphi\left(\begin{pmatrix} r_1 & s_1 \\ 0 & t_1 \end{pmatrix} + \begin{pmatrix} r_2 & s_2 \\ 0 & t_2 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} r_1 + r_2 & s_1 + s_2 \\ 0 & t_1 + t_2 \end{pmatrix}\right) = (r_1 + r_2, t_1 + t_2)$$
$$\varphi\left(\begin{pmatrix} r_1 & s_1 \\ 0 & t_1 \end{pmatrix}\right) + \varphi\left(\begin{pmatrix} r_2 & s_2 \\ 0 & t_2 \end{pmatrix}\right) = (r_1, t_1) + (r_2, t_2) = (r_1 + r_2, t_1 + t_2)$$

thus φ is a ring homomorphism, and it is surjective since for any $r, t \in \mathbb{R} \times \mathbb{R}$ we have:

$$\varphi\left(\begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix}\right) = (r, t)$$

and:

$$\ker(\varphi) = \left\{ \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} : \varphi\left(\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}\right) = (0, 0) \right\} = \left\{ \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} : r = 0 = t, s \in \mathbb{R} \right\} = I,$$

thus by the First Isomorphism Theorem, we have that:

$$U/I = U/\ker(\varphi) \cong \operatorname{img}(\varphi) = \mathbb{R} \times \mathbb{R}.$$

Exercise 5

Let R be a ring, consider $M_n(R)$. We want to prove that for any ideal $J \subset M_n(R)$ there exists an ideal $I \subset R$ such that $J = M_n(I)$.

To prove this, given $J \subset M_n(R)$ an ideal, define:

$$I = \{a \in R : a \text{ is an entry of some matrix } A \in J\}.$$

We will now prove that if $A \in J$, then for every entry a of A the matrix that has a in the position (i, j) and the rest zeroes also belongs to J. For this, note that if $A_{s,t}$ is the matrix with 1 in the position (s, t) and the rest zeroes, then for any matrix $M \in M_n(R)$ we have that $A_{s,t}M$ is the matrix that has in the s-th row the t-th row of M, and the rest zeroes. Moreover, $MA_{i,j}$ is the matrix that has in the j-th row the i-th row of M, and the rest zeroes. That is, the matrix $A_{s,t}MA_{i,j}$ chooses the entry (t, i) of M and puts it as the entry (s, j), leaving the rest as zeroes. Since $A_{i,j} \in M_n(R)$ for every $i, j \in \{1, \ldots, n\}$ and J is an ideal, we have that $A_{s,t}AA_{i,j} \in J$ for every $A \in J$ (in particular $A_{i,i}AA_{j,j} \in J$ has $a_{i,j}$ in the position (i, j) and zeroes elsewhere).

Thus, consider $a \in I$. This means that there is a matrix $A \in J$ with a in some entry, say (t, i). By the above, we have that $A_{1,t}AA_{i,1} \in J$ is the matrix that has a in the position $(1, 1)$ and zeroes elsewhere. Now, for any $r \in R$, consider $M(r)$ the matrix having a in the position $(1, 1)$ and zeroes elsewhere. Now $A_{1,t}AA_{i,1}M(r) \in J$ and $M(r)A_{1,t}AA_{i,1} \in J$ because J is an ideal, and these matrices are $M(ar)$ and $M(ra)$ respectively, that is, it they have ar and ra (respectively) in the position $(1, 1)$ and zeroes elsewhere. In particular, $ar, ra \in I$ and I is an ideal in R.

Moreover, we have $J = M_n(I)$ as follows. If $A \in J$, then the entries of A belong to I by definition and thus $A \in M_n(I)$. If $A \in M_n(I)$ this means that the entries of A belong to I thus for each $a_{i,j}$ with $i, j \in \{1, \ldots, n\}$ there is at least a matrix in J having $a_{i,j}$ as an entry. Now A can be written as the sum of the matrices having $a_{i,j}$ in the position (i, j) and zeroes elsewhere, and since we proved that these matrices belong to J above, and J is an ideal, this sum belongs to J hence $A \in J$.

Exercise 6

We show that $M_n(\mathbb{R})$ is simple. For this, suppose $J \subset M_n(\mathbb{R})$ is a two sided ideal, then by the Exercise 5 above we have that there exists $I \subset \mathbb{R}$ an associated ideal such that $J = M_n(I)$. However, since $\mathbb{R}$ is a field, we have that $I = \{0\}, \mathbb{R}$. In the first case, $J = \{(0)_{i,j}\}$ and in the second $J = M_n(\mathbb{R})$, hence J is never a proper ideal.

Consider now the one sided ideal I formed by the matrices that are all zeroes except the last column. We clearly have that for $a_{i,j}, r_i, s_j \in \mathbb{R}$ for $i, j \in \{1, \ldots, n\}$:

$$\begin{pmatrix} 0 & \cdots & 0 & r_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & r_n \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & s_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & s_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & r_1 s_n \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & r_n s_n \end{pmatrix} \in I$$

$$\begin{pmatrix} 0 & \cdots & 0 & r_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & r_n \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & s_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & s_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & r_1 + s_n \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & r_n + s_n \end{pmatrix} \in I$$

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & r_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & r_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & t_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & t_n \end{pmatrix} \in I$$

$$\begin{pmatrix} 0 & \cdots & 0 & r_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & r_n \end{pmatrix} \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} *_{1,1} & \cdots & *_{1,n} \\ \vdots & & \vdots \\ *_{n,1} & \cdots & *_{n,n} \end{pmatrix} \notin I$$

where $t_i \in \mathbb{R}$ is the corresponding multiplication of elements $a_{u,v}$ and r_k for every $u, v, i \in \{1, \ldots, n\}$, and $*_{i,j}$ for $i, j \in \{1, \ldots, n\}$ denotes an entry that may not always be zero. Thus I is only a one sided ideal. Moreover, notice that it is non empty since taking $r_j = 1$ for $j \in \{1, \ldots, n\}$ we obtain a non zero element belonging to I. This I is as desired.