Algebra II - Homework 1

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Exercise 1

Let $\mathcal{C}$ be a category and $f : A \longrightarrow B$ an isomorphism with inverse $g : B \longrightarrow A$. To prove that g is unique, let $h : B \longrightarrow A$ be another inverse of f, that is, a morphism with the same properties as g. Now:

$$g = g \circ (f \circ h) = (g \circ f) \circ h = h,$$

as desired.

Exercise 2

1. In the category of sets, prove that a morphism $f : A \longrightarrow B$ is a monomorphism if and only if it is injective.

   $\Rightarrow$) Let f be a monomorphism. Suppose $x, y \in A$ with $f(x) = f(y)$, let $Z = \{0\}$ and define the functions $g_x : Z \longrightarrow A$ as $g_x(0) = x$ and $g_y : Z \longrightarrow A$ as $g_x(0) = y$. Since $f \circ g_x(0) = f(x) = f(y) = f \circ g_y(0)$ this means $f \circ g_x = f \circ g_y$, hence since f is a monomorphism we get that $g_x = g_y$, thus $x = g_x(0) = g_y(0) = y$ and f is injective.

   $\Leftarrow$) Let f be injective. Suppose we have $g_1, g_2 : Z \longrightarrow A$ with $f \circ g_1 = f \circ g_2$. If there is $x \in A$ with $g_1(x) \neq g_2(x)$, by the injectivity of f we have that $f \circ g_1(x) \neq f \circ g_2(x)$. However, this is a contradiction. Hence, there is no such $x \in A$, that is, $g_1(x) = g_2(x)$ for every $x \in A$, that is, $g_1 = g_2$ and f is a monomorphism.

2. In the category of sets, prove that a morphism $f : A \longrightarrow B$ is an epimorphism if and only if it is surjective.

   $\Rightarrow$) Let f be an epimorphism. Let $y \in B$ and suppose there is no $x \in A$ with $f(x) = y$. Let $Z = \{0, 1\}$ and define the functions $g_1 : B \longrightarrow Z$ as $g_1(y) = 1$, $g_1(b) = 0$ when $y \neq b \in B$ and $g_2 : B \longrightarrow Z$ as $g_2(b) = 0$ for every $b \in B$. Now we have that $f(A) \subset B \setminus \{y\}$ hence $g_1 \circ f = g_2 \circ f$, meaning that $g_1 = g_2$ since f is an epimorphism. However, this is a contradiction since $g_1(y) \neq g_2(y)$. Thus there is no such $y \in B$, that is, for every $y \in B$ there is an $x \in A$ with $f(x) = y$, that is, f is surjective.

   $\Leftarrow$) Let f be surjective. Suppose we have $g_1, g_2 : B \longrightarrow Z$ with $g_1 \circ f = g_2 \circ f$. Let $y \in B$, then by surjectivity of f there is an $x \in A$ with $f(x) = y$. Now:

   $$g_1(y) = g_1 \circ f(x) = g_2 \circ f(x) = g_2(y),$$

   hence $g_1 = g_2$ and f is an epimorphism.

3. We show that in the category of rings with unity, the inclusion $\phi : \mathbb{Z} \longrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism.

   Suppose we have $g_1, g_2 : A \longrightarrow \mathbb{Z}$ morphisms with $\phi \circ g_1 = \phi \circ g_2$. For every $x \in A$, the function ϕ let us look $g_1(x)$ and $g_2(x)$ as elements in $\mathbb{Q}$. We have: $g_1(x) = \phi \circ g_1(x) = \phi \circ g_2(x) = g_2(x)$ in $\mathbb{Q}$. However, since we know that $g_1(x), g_2(x) \in \mathbb{Z}$ and that ϕ is injective, we have that $g_1(x) = g_2(x)$ hence $g_1 = g_2$ and ϕ is a monomorphism.

   Suppose we have $g_1, g_2 : \mathbb{Q} \longrightarrow A$ morphisms with $g_1 \circ \phi = g_2 \circ \phi$. For every $p/q \in \mathbb{Q}$ we have:

   $$\begin{aligned} g_1(p/q) &= g_1(p)g_1(q^{-1}) = (g_1 \circ \phi(p))(g_1 \circ \phi(q))^{-1} \\ &= (g_2 \circ \phi(p))(g_2 \circ \phi(q))^{-1} = g_2(p)g_2(q^{-1}) = g_2(p/q), \end{aligned}$$

where we have used that g_1 and g_2 are morphisms in the first, second, fourth and fifth equalities, the condition in the third and the fact that since $p, q \in \mathbb{Z}$, we have that by the injectivity of ϕ we can think of them as $p = \phi(p)$ and $q = \phi(q)$. Hence $g_1 = g_2$ and ϕ is an epimorphism.

Exercise 3

Consider R a commutative ring and $X = \mathrm{Spec}(R)$ his spectrum. We define $Z(E) = \{P \in X : E \subset P\}$ for any $E \subset R$. We prove:

1. Let A be the ideal generated by E. Then $Z(E) = Z(A)$:

   $\supseteq$) Let $P \in Z(A)$, that is, $P \in X$ with $A \subset P$. Then $E \subset A \subset P$ thus $P \in Z(E)$.

   $\subseteq$) Let $P \in Z(E)$, that is, $P \in X$ with $E \subset P$. Since P is an ideal containing E and A is the smallest ideal containing E, we must have that $A \subset P$, hence $P \in Z(A)$.

2. Prove $Z(0) = X$ and $Z(1) = \emptyset$. Notice that a every ideal contains the element 0 and that prime ideals are proper, that is, they are not the whole ring R thus they do not contain the element 1. Thus:

   $$Z(0) = \{P \in X : \{0\} \subset P\} = \{P \in X\} = X, \quad Z(1) = \{P \in X\{1\} \subset P\} = \{\} = \emptyset.$$

3. Let $\{E_i\}_{i\in I}$ be a family of subsets of R, then:

   $$\begin{aligned} Z\left(\bigcup_{i\in I} E_i\right) &= \left\{P \in X : \bigcup_{i\in I} E_i \subset P\right\} = \{P \in X : E_i \subset P, \forall i \in I\} \\ &= \{P \in X : P \in Z(E_i), \forall i \in I\} = \bigcap_{i\in I} Z(E_i). \end{aligned}$$

4. Let A, B, C be ideals, prove that $Z(A \cap B) = Z(AB) = Z(A) \cup Z(B)$. First, we note that in virtue of the first point above, we can consider AB just as a set. Now:

   $Z(A \cap B) \subseteq Z(AB)$: Let $P \in Z(A \cap B)$, that is, $P \in X$ with $A \cap B \subset P$. Let $ab \in AB$, that is, $a \in A$, $b \in B$. Since A and B are ideals, we have that $ab \in A$ and $ab \in B$, hence $ab \in A \cap B$ and $AB \subset A \cap B \subset P$, thus $P \in Z(AB)$.

   $Z(AB) \subseteq Z(A) \cup Z(B)$: Let $P \in Z(AB)$, that is, $P \in X$ with $AB \subset P$. Since P is a prime ideal, we automatically have that either $A \subset P$ or $B \subset P$, hence either $P \in Z(A)$ or $P \in Z(B)$ respectively, meaning that $P \in Z(A) \cup Z(B)$.

   $Z(A) \cup Z(B) \subseteq Z(A \cap B)$: Let $P \in Z(A) \cup Z(B)$, that is, $P \in X$ with either $A \subset P$ or $B \subset P$. Using $A \cap B \subset A \subset P$ or $A \cap B \subset B \subset P$ respectively, we obtain that $P \in Z(A \cap B)$.

5. For the set $\tau = \{Z(E) : E \subset R\}$ to define the closed sets on X, that is, to be a topology on X we need three properties that follow immediately applying what we have proven above:

   (a) $\emptyset = Z(1) \in \tau$, $X = Z(0) \in \tau$.
   (b) Given $A, B \subset R$, we have $Z(A) \cup Z(B) = Z(A \cap B) \in \tau$.
   (c) Given $\{E_i\}_{i\in I}$ a family of subsets of R, we have $\cap_{i\in I} Z(E_i) = Z(\cup_{i\in I} E_i) \in \tau$.

Exercise 4

Consider the particular case of $X = \mathrm{Spec}(\mathbb{Z})$, let $\tau = \{Z(E) : E \subset \mathbb{Z}\}$.

1. Prove that $X = \{(p) : p \text{positive prime}\} \cup \{(0)\}$

   $\supseteq$) Clearly (0) is a prime ideal since $\mathbb{Z}$ is a domain. Moreover, if we have $a, b \in \mathbb{Z}$ with $ab \in (p)$, then p divides ab and since p is prime, p must divide a or b, that is, $a \in (p)$ or $b \in (p)$ respectively, meaning that (p) is a prime ideal.

   $\subseteq$) Let $P \subset \mathbb{Z}$ be a prime ideal, consider $p \in P$ the smallest number in P. For any $a \in P$, apply the division algorithm and obtain that $a = pq + c$ for certain $q, c \in \mathbb{Z}$ with $c < p$. Now since $c = a - pq \in P$ because P is an ideal and p is the smallest, we must have $c = 0$, hence $P = (p)$. There are two possibilities, $p = 0$ or p positive. Clearly (0) is prime. In the second case, since (p) must be prime, when we have $a, b \in \mathbb{Z}$ with $ab \in (p)$, that is, there exists $c \in \mathbb{Z}$ with $ab = pc$, we must have that $a \in (p)$ or $b \in (p)$, that is there exist $f \in \mathbb{Z}$ such that $a = pf$ or $g \in \mathbb{Z}$ such that $b = pg$. Summing up, when p divides ab then p divides a or b, that is, p is prime.

2. To prove that for a positive prime p we have $\overline{\{(p)\}} = \{(p)\}$ it is enough to prove that $\{(p)\} \in \tau$. We claim that $Z((p)) = \{(p)\}$.

   $\supseteq$) We know that $(p) \in X$ and $(p) \subset (p)$, hence $(p) \in Z((p))$.

   $\subseteq$) Let $P \in X$, by the above, we know that $P = (q)$ for certain prime $q \in \mathbb{Z}$, with $(p) \subset P$. Thus $(p) \subset (q)$, meaning that q divides p. However, p is prime, thus q is either 1 or p. Since prime ideals are proper, we must have $1 \notin (q)$ hence $q = p$.

3. To find $\overline{\{(0)\}}$, we note that for every $E \subset R$ the ideal generated by E has (0) as a subset, hence $(0) \in Z(E)$. Since $\overline{\{(0)\}}$ is the intersection of all the closed sets that contain (0), we have $\overline{\{(0)\}} = \cap_{E \subset R} Z(E) = Z(\cup_{E \subset R} E) = Z(R) = Z(1) = X$.

Exercise 5

Let $\mathcal{R}$ the category of commutative rings with unity and $\mathcal{T}$ the category of topological spaces. Define the functor:

$$\begin{array}{rcrcl} F & : & \mathcal{R} & \longrightarrow & \mathcal{T} \\ & & R & \longmapsto & \mathrm{Spec}(R) \end{array}$$

such that given $f : R \to S$ it assigns the map:

$$\begin{array}{rcccc} F(f) & : & \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \\ & & P & \longmapsto & f^{-1}(P) \end{array}$$

We note that $F(f)$ is well defined, that is, $f^{-1}(P)$ is a prime ideal of R:

1. Let $a, b \in f^{-1}(P)$, $r \in R$, then using that f is a homomorphism we obtain $f(a-b) = f(a) - f(b) \in P$ and $f(ra) = f(r)f(a) \in P$ since P is an ideal and it is prime. This means that $f^{-1}(P)$ is an ideal.

2. Let $a, b \in R$ such that $ab \in f^{-1}(P)$, then $P \ni f(ab) = f(a)f(b)$ and since P is prime, this means that either $f(a) \in P$ or $f(b) \in P$, that is $a \in f^{-1}(P)$ or $b \in f^{-1}(P)$, meaning that $f^{-1}(P)$ is prime.

We check that $F(f)$ is a morphism in $\mathcal{T}$, that is, $F(f)$ is continuous. For this, it is enough to see that the preimage of closed sets is closed: let $E \subset R$, we have that:

$$\begin{aligned} F(f)^{-1}(Z(E)) &= \{P \in \mathrm{Spec}(S) : f^{-1}(P) \in Z(E)\} = \{P \in \mathrm{Spec}(S) : E \subset f^{-1}(P)\} \\ &= \{P \in \mathrm{Spec}(S) : f(E) \subset P\} = Z(f(E)), \end{aligned}$$

which is closed in $\mathrm{Spec}(S)$, meaning that $F(f)$ is continuous as desired.

The only thing left to check is that F satisfies the two required contravariant properties on the morphisms:

1. For every $P \in \mathrm{Spec}(R)$ we have $F(\mathrm{id}_R)(P) = \mathrm{id}_R^{-1}(P) = P = \mathrm{id}_{\mathrm{Spec}(R)}(P)$, thus $F(\mathrm{id}_R) = \mathrm{id}_{\mathrm{Spec}(R)} = \mathrm{id}_{F(R)}$.

2. Let $f : R \longrightarrow S$ and $g : S \longrightarrow T$ homomorphisms of rings, then for every $P \in \mathrm{Spec}(T)$ we have $F(g \circ f)(P) = (g \circ f)^{-1}(P) = f^{-1} \circ g^{-1}(P) = f^{-1}(F(g)(P)) = F(f) \circ F(g)(P)$, thus $F(g \circ f) = F(f) \circ F(g)$.