Algebra II - Homework 3

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Exercise 1

1. Show that $S^{-1}M$ is a $S^{-1}R$ module. First, notice that $(S^{-1}M,+)$ is an abelian group since:

   (a) Given $m_1/s_1, m_2/s_2 \in$ we have: $s_2m_1 + s_1m_2 \in M, s_1s_2 \in S$ and:

   $$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2m_1 + s_1m_2}{s_1s_2} = \frac{s_1m_2 + s_2m_1}{s_1s_2} = \frac{m_2}{s_2} + \frac{m_1}{s_1} \in S^{-1}M.$$

   (b) We have that $0/1 \in S^{-1}M$ is the identity element, for every $m/s \in S^{-1}M$:

   $$\frac{m}{s} + \frac{0}{1} = \frac{m+0}{s} = \frac{m}{s}.$$

   (c) Given $m/s \in S^{-1}M$, we have $(-m)/s \in S^{-1}M$ is its inverse:

   $$\frac{m}{s} + \frac{-m}{s} = \frac{sm - sm}{s^2} = \frac{0}{s^2} = \frac{0}{1}.$$

   Now, we define the multiplication on $S^{-1}M$ by scalars $S^{-1}R$ as:

   $$\begin{array}{ccc} S^{-1}R \times S^{-1}M & \longrightarrow & S^{-1}M \\ \left(\frac{r}{t}, \frac{m}{s}\right) & \longmapsto & \frac{rm}{ts} \end{array}$$

   and equipped with this, we notice that for every $m_1/s_1, m_2/s_1, m/s \in S^{-1}M$ and $r_1/t_1, r_2/t_2, r/t \in S^{-1}R$ we have:

   (a) $\frac{r}{t}(\frac{m_1}{s_1} + \frac{m_2}{s_2}) = \frac{r}{t}\frac{s_2m_1+s_1m_2}{s_1s_2} = \frac{rs_2m_1+rs_1m_2}{ts_1s_2} = \frac{trs_2m_1+trs_1m_2}{ts_1ts_2} = \frac{rm_1}{ts_1} + \frac{rm_2}{ts_2} = \frac{r}{t}\frac{m_1}{s_1} + \frac{r}{t}\frac{m_2}{s_2}$.

   (b) $(\frac{r_1}{t_1} + \frac{r_2}{t_2})\frac{m}{s} = \frac{t_2r_1+t_1r_2}{t_1t_2}\frac{m}{s} = \frac{t_2r_1m+t_1r_2m}{t_1t_2s} = \frac{t_2sr_1m+t_1sr_2m}{t_1st_2s} = \frac{r_1m}{t_1s} + \frac{r_2m}{t_2s} = \frac{r_1}{t_1}\frac{m}{s} + \frac{r_2}{t_2}\frac{m}{s}$.

   (c) $(\frac{r_1}{t_1}\frac{r_2}{t_2})\frac{m}{s} = \frac{r_1r_2}{t_1t_2}\frac{m}{s} = \frac{r_1r_2m}{t_1t_2s} = \frac{r_1}{s_1}\frac{r_2m}{t_2s} = \frac{r_1}{t_1}(\frac{r_2}{t_2}\frac{m}{s})$.

   (d) $\frac{1}{1}\frac{m}{s} = \frac{1m}{1s} = \frac{m}{s}$.

   And the axioms of $S^{-1}M$ being an $S^{-1}R$ module are proven, as desired.

2. Given $\phi : M \longrightarrow N$ an R module homomorphism we define $S^{-1}\phi : S^{-1}M \longrightarrow S^{-1}N$ via $S^{-1}\phi(m/s) = \phi(m)/s$ for $m/s \in S^{-1}M$. Although not explicitly asked, it is clear that $S^{-1}\phi$ is an $S^{-1}R$ module homomorphism since for $m_1/s_1, m_2/s_2, m/s \in S^{-1}M$ and $r/t \in S^{-1}R$:

   (a) $S^{-1}\phi(\frac{m_1}{s_2} + \frac{m_2}{s_2}) = S^{-1}\phi(\frac{s_2m_1+s_1m_2}{s_1s_2}) = \frac{\phi(s_2m_1+s_1m_2)}{s_1s_2} = \frac{s_2\phi(m_1)+s_1\phi(m_2)}{s_1s_2} = \frac{\phi(m_1)}{s_1} + \frac{\phi(m_2)}{s_2} = S^{-1}\phi(\frac{m_1}{s_1}) + S^{-1}\phi(\frac{m_2}{s_2})$,

   (b) $S^{-1}\phi(\frac{r}{t}\frac{m}{s}) = S^{-1}\phi(\frac{rm}{ts}) = \frac{\phi(rm)}{ts} = \frac{r\phi(m)}{ts} = \frac{r}{t}\frac{\phi(m)}{s} = \frac{r}{t}S^{-1}\phi(\frac{m}{s})$.

We now show that $S^{-1}(\cdot) : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(S^{-1}R)$ is a covariant functor:

(a) To each object $M \in \mathrm{Mod}(R)$ we have $S^{-1}M \in \mathrm{Mod}(S^{-1}R)$ by the above.

(b) To each morphism $\phi : M \longrightarrow N$ in $\mathrm{Mod}(R)$ we have $S^{-1}\phi : S^{-1}M \longrightarrow S^{-1}N$ a morphism in $\mathrm{Mod}(S^{-1}R)$ by the above.

(c) For every $m/s \in S^{-1}M$ we have: $S^{-1}(\mathrm{id}_M)(\frac{m}{s}) = \frac{\mathrm{id}_M(m)}{s} = \frac{m}{s} = \mathrm{id}_{S^{-1}M}(\frac{m}{s})$ thus $S^{-1}(\mathrm{id}_M) = \beth_{S^{-1}M}$.

(d) Given any morphisms $\psi : M \longrightarrow N$ and $\phi : N \longrightarrow K$, for every $m/s \in S^{-1}M$ we have: $S^{-1}(\phi \circ \psi)(\frac{m}{s}) = \frac{\phi\circ\psi(m)}{s} = S^{-1}\phi(\frac{\psi(m)}{s}) = S^{-1}\phi(S^{-1}\psi(\frac{m}{s}))$ thus $S^{-1}(\phi \circ \psi) = S^{-1}\phi \circ S^{-1}\psi$.

And the axioms of $S^{-1}(\cdot)$ being a functor are proven, as desired.

3. Show that $S^{-1}(\cdot)$ is exact. For this, let $N \xrightarrow{\phi} M \xrightarrow{\psi} K$ be an exact sequence, that is, $\mathrm{im}(\phi) = \ker(\psi)$, we want to prove that $S^{-1}M \xrightarrow{S^{-1}\phi} S^{-1}M \xrightarrow{S^{-1}\psi} S^{-1}K$ is exact, that is, $\mathrm{im}(S^{-1}\phi) = \ker(S^{-1}\psi)$.

   $\subseteq$) Let $m/s \in \mathrm{im}(S^{-1}\phi)$, that is, there is $n/t \in S^{-1}N$ with $\phi(n)/t = S^{-1}\phi(n/t) = m/s$. Hence $S^{-1}\psi(m/s) = S^{-1}\psi(\phi(n)/t) = \psi \circ \phi(n)/t = 0/t = 0/1$ since $\phi(n) \in \mathrm{im}(\phi) = \ker(\psi)$ thus $\psi(\phi(n)) = 0$, that is, $m/s \in \ker(S^{-1}\psi)$.

   $\supseteq$) Let $m/s \in \ker(S^{-1}\psi)$, that is, $0 = S^{-1}\psi(m/s) = \psi(m)/s$, hence there is $t \in S$ with $t\psi(m) = 0$, and since ψ is a morphism this happens if and only if $\phi(tm) = 0$. Hence $tm \in \ker(\psi) = \mathrm{im}(\phi)$ and there is $n \in N$ with $\phi(n) = tm$, with $n/ts \in S^{-1}N$. Thus $S^{-1}\phi(n/ts) = \phi(n)/ts = tm/ts = m/s$ meaning that $m/s \in \mathrm{im}(S^{-1}\phi)$.

Exercise 2

In the notation as above, let $K = S^{-1}R$.

1. Show that there is a ring homomorphism $\phi : R \longrightarrow K$ such that $\phi(r) = r/1$, and this is injective if and only if S does not contain 0 nor zero divisors. By simply defining:

$$\begin{array}{ccccc} \phi & : & R & \longrightarrow & K \\ & & r & \longmapsto & \frac{r}{1} \end{array}$$

   we note that to each $r \in R$ corresponds a single $r/1 \in K$, hence this is a well defined function. This function is a ring homomorphism since for $r_1, r_2 \in R$ we have:

   (a) $\phi(r_1 + r_2) = \frac{r_1+r_2}{1} = \frac{r_1}{1} + \frac{r_2}{1} = \phi(r_1) + \phi(r_2)$,
   (b) $\phi(r_1 r_2) = \frac{r_1 r_2}{1} = \frac{r_1}{1}\frac{r_2}{1} = \phi(r_2)\phi(r_2)$,
   (c) $\phi(1) = 1/1$ the unit element in K.

   Notice that for $r, s \in R$ we have:

$$\phi(r) = \phi(s) \iff \frac{r}{1} = \frac{s}{1} \iff \exists t \in S \text{ with } t(r-s) = 0,$$

   hence for proving that ϕ is injective if and only if S does not contain 0 nor zero divisors:

   $\Rightarrow$) If S contains 0 then using $t = 0$ above we clearly have that any two elements $r, s \in R$ have the same image, thus ϕ is not injective. If S contains a zero divisor v, say that $vu = 0$ for certain non zero $u \in R$, then taking $t = v$ above, we have that $0 = tu = t(u - 0)$ thus $\phi(u) = \phi(0)$ with $u \neq 0$, hence ϕ is not injective.

   $\Leftarrow$) Suppose ϕ is not injective, this means that there are different $r, s \in R$ and $t \in S$ with $\phi(r) = \phi(s)$, that is, $t(r - s) = 0$. If $t = 0$ then $0 \in S$ and if $t \neq 0$ then since $r - s \neq 0$ we have that $t \in S$ is a zero divisor.

2. Let M be an R module, we define the natural multiplication on $S^{-1}M$ by elements of R to make it an R-module. Since we are not asked to prove this, we will take it for granted, but notice how this is a particular case of Exercise 1 by taking $S = \{1\}$ and thus the proof is exactly the same as we have exposed above, by simply replacing the elements in $S^{-1}R$ by elements of the form $r/1$ with $r \in R$. We want to show that the function:

$$\begin{array}{ccccc} \psi & : & M & \longrightarrow & S^{-1}M \\ & & m & \longmapsto & \frac{m}{1} \end{array}$$

   is an R-module homomorphism. For this, consider $m, n \in M$ and $r \in R$, we have:

   (a) $\psi(m + n) = \frac{m+n}{1} = \frac{m}{1} + \frac{n}{1} = \psi(m) + \psi(n)$,
   (b) $\psi(rm) = \frac{rm}{1} = \frac{r}{1}\frac{m}{1} = r\frac{m}{1} = r\psi(m)$.

Finally, to compute $\ker(\psi)$ we realize that for $m \in M$ we have $\psi(m) = 0/1$ if and only if $m/1 = 0/1$, that is, whenever there is an $t \in S$ with $tm = 0$. Hence:

$$\ker(\psi) = \{m \in M : \exists t \in S \text{ with } tm = 0\}.$$

Exercise 3

Let R be a commutative ring, M an R-module, we define for a prime ideal $P \subset R$ the set $S_P = R \setminus P$ and $R_P = S_P^{-1}R$, $M_P = S_P^{-1}M$. We are asked to show that the following are equivalent:

1. $M = 0$,
2. $M_P = 0$ for all prime ideals $P \subset R$,
3. $M_N = 0$ for all maximal ideals $N \subset R$.

1. $\Rightarrow$)2. Let $P \subset R$ be a prime ideal. If $M = 0$ then clearly $0/r = 0/1$ for every $r \in S_P$, thus $M_P = \{\frac{0}{1}\} = 0$.

2. $\Rightarrow$)1. Let $N \subset R$ be a maximal ideal. Since R is commutative, N is a prime ideal, hence $M_N = 0$.

2. $\Rightarrow$)1. Suppose $M \neq 0$, pick $m \in M$ non zero. Consider the set $\{r \in R : rm = 0\}$, which is clearly an ideal of R by the properties of the multiplication on M by R as a module (that is, it is an additive group and $R\{r \in R : rm = 0\} \subset \{r \in R : rm = 0\}$). Since every ideal is contained in a maximal ideal, there is a maximal ideal N with $\{r \in R : rm = 0\} \subset N$. However, now $m/1 \neq 0/1$ in M_N, since if $m/1 = 0/1$ in M_N this means there is an element $t \in S_N$ with $tm = 0$, but now $t \in \{r \in R : rm = 0\}$, which is a contradiction. Hence $M_N \neq 0$ and by contrapositive we are done.

Exercise 4

Let R be a ring (we assume with unity) and let $R' = \text{Hom}(R, R)$. Show that we have $R' \cong R^{op}$ where R^{op} is the opposite ring of R. We will prove the equivalent statement that $R \cong \text{Hom}(R^{op}, R^{op})$. For this, we define:

$$\begin{array}{cccc} \phi & : & R & \longrightarrow & \text{Hom}(R^{op}, R^{op}) \\ & & a & \longmapsto & f_a \end{array}, \quad \text{with} \quad \begin{array}{cccc} f_a & : & R^{op} & \longrightarrow & R^{op} \\ & & r & \longmapsto & r * a \end{array}$$

which is clearly well defined since f_a is a multiplication by an element of the ring which is always a morphism of rings, thus:

1. ϕ is a morphism of rings, for $a, b \in R$ and $r \in R^{op}$ we have:
   (a) $\phi(a+b)(r) = r*(a+b) = (a+b)r = ar+br = r*a+r*b = \phi(a)(r)+\phi(b)(r)$ so $\phi(a+b) = \phi(a) + \phi(b)$,
   (b) $\phi(ab)(r) = r*(ab) = abr = a(r*b) = (r*b)*a = \phi(a)(r*b) = \phi(a)(\phi(b)(r))$ so $\phi(ab) = \phi(a) \circ \phi(b)$,
   (c) $\phi(1)(r) = r*1 = 1r = r = \text{id}_R(r)$.
2. ϕ is injective: suppose we have $\phi(a)(r) = \phi(b)(r)$, this happens if and only if $ar = r*a = r*b = br$ thus in particular taking $r = 1$ we obtain $a = b$ and injectivity.
3. ϕ is surjective: suppose we have a function $f : R^{op} \longrightarrow R^{op}$, let $f(1) \in R^{op}$, we claim that $\phi(f(1)) = f$. For this, notice that $\phi(f(1))(r) = r * f(1) = f(r1) = f(r)$ as desired, where we used that f is a morphism.

Hence ϕ is an isomorphism of rings, as desired.

Exercise 5

Let K be a field, V a finite dimensional vector space over K.

1. Show that V satisfies the ascending chain condition. Suppose we have $\{V_i\}_{i=0}^{\infty}$ a sequence of subspaces such that $V_0 \subset V_1 \subset \cdots$. Since V is finite dimensional, suppose $\dim(V) = n$. Hence $d_i = \dim(V_i) \leq n$ for every i. Moreover since subspaces of the same dimension are are isomorphic, we can consider the chain above up to isomorphism to obtain $V_{i_0} \subsetneq V_{i_1} \subsetneq \cdots$. This induces a strictly increasing sequence of natural numbers $d_{i_0} < d_{i_1} < \cdots$. Since we have $0 \leq d_i \leq n$, this is a strictly increasing sequence of natural numbers bounded above by n, hence it must stabilize, say on d_{i_j}. Thus taking $N = i_j$, for $i \geq N$ we must have $V_i = V_N$, as desired.

2. Show that V satisfies the descending chain condition. The argument is exactly the same, but changing the sense of the inclusions. Suppose we have $\{V_i\}_{i=0}^{\infty}$ a sequence of subspaces such that $V_0 \supset V_1 \supset \cdots$. Since V is finite dimensional, suppose $\dim(V) = n$. Hence $d_i = \dim(V_i) \leq n$ for every i. Moreover since subspaces of the same dimension are are isomorphic, we can consider the chain above up to isomorphism to obtain $V_{i_0} \supsetneq V_{i_1} \supsetneq \cdots$. This induces a strictly decreasing sequence of natural numbers $d_{i_0} > d_{i_1} > \cdots$. Since we have $0 \leq d_i \leq n$, this is a strictly decreasing sequence of natural numbers bounded below by 0, hence it must stabilize, say on d_{i_j}. Thus taking $N = i_j$, for $i \geq N$ we must have $V_i = V_N$, as desired.