Algebra II - Homework 4

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March 3rd, 2017

Exercise 1

Let $R = \mathbb{Z}[\sqrt{-6}]$.

1. Show that the ideal $A = (2, \sqrt{-6})$ is not principal. For this, we first define the function $N : R \longrightarrow \mathbb{N}$ as $N(a + b\sqrt{-6}) = a^2 + 6 + b^2$ for $a, b \in \mathbb{Z}$. Notice how for $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ we have:

$$\begin{aligned} N((a_1 + b_1\sqrt{-6})(a_2 + b_2\sqrt{-6})) &= N((a_1a_1 - 6b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{-6}) \\ &= (a_1a_1 - 6b_1b_2)^2 + 6(a_1b_2 + b_1a_2)^2 \\ &= a_1^2a_2^2 + 6b_1^2a_2^2 + 6a_1^2b_2^2 + 36b_1^2b_2^2 \\ &= (a_1^2 + 6b_1^2)(a_2^2 + 6b_2^2) \\ &= N(a_1 + b_1\sqrt{-6})N(a_2 + b_2\sqrt{-6}), \end{aligned}$$

   which means that for $r, s \in \mathbb{Z}[\sqrt{-6}]$ we have $N(rs) = N(r)N(s)$. Once we have this, suppose that $(2, \sqrt{-6}) = (a)$ for certain $a_1 + a_2\sqrt{-6} = a \in \mathbb{Z}[\sqrt{-6}]$, we want to find a contradiction. This means that there are $r, s \in \mathbb{Z}[\sqrt{-6}]$ such that:

$$\left.\begin{array}{r} 2 = ar \\ \sqrt{-6} = as \end{array}\right\} \Longrightarrow \left.\begin{array}{c} 4 = N(2) = N(ar) = N(a)N(r) \\ 6 = N(\sqrt{-6}) = N(as) = N(a)N(s) \end{array}\right\} \Longrightarrow N(a) |\gcd(4, 6) = 2$$

   because everything is in $\mathbb{N}$, thus we have either:

   (i) $N(a) = 1$ meaning that $a = \pm 1$. Hence $1 \in A$ and there are $u, v \in \mathbb{Z}[\sqrt{-6}]$, say $u = u_1 2 + u_2\sqrt{-6}$, $v = v_1 2 + v_2\sqrt{-6}$, such that:

$$1 = u2 + v\sqrt{-6} = u_1 2 - 6v_2 + u_2 2\sqrt{-6} + v_1\sqrt{-6} \Longrightarrow 1 = 2(u_1 - 3v_2)$$

   which is impossible since $u_1 - 3v_2 \in \mathbb{Z}$, thus we have a contradiction.

   (ii) $N(a) = 2$ meaning that $2 = a_1^2 + 6a_2^2$. If $a_1 = 0$, we have that $2 > 0$ if $a_2 = 0$ and $2 < 6a_2^2$ for every $a_2 \neq 0$, a contradiction in either case. If $|a_1| = 1$ we have that $2 > 1$ if $a_2 = 0$ and $2 < 1 + 6a_2^2$ for every $a_2 \neq 0$, a contradiction in either case. If $|a_1| > 1$ we have that $2 < a_1^2 + 6a_2^2$ for every $a_2 \in \mathbb{Z}$, a contradiction.

   Thus we found a contradiction in every possible outcome. This means that there does not exist such an $a \in \mathbb{Z}[\sqrt{-6}]$ and A is not principal.

2. Show that A is projective as an R-module. Suppose we have R-module homomorphisms $f : N \longrightarrow M$ and $g : M \longrightarrow A$ such that $0 \longrightarrow N \longrightarrow M \longrightarrow A \longrightarrow 0$ is an exact sequence. Since $g : M \longrightarrow A$ is surjective, there are elements $\alpha, \beta \in M$ such that $g(\alpha) = 2$ and $g(\beta) = \sqrt{-6}$. Consider the function $\psi : A \longrightarrow M$ defined by $\psi(r2 + s\sqrt{-6}) = r\alpha + s\beta$ for every $r, s \in \mathbb{Z}[\sqrt{-6}]$. First, note that this is an R-module homomorphism, since for $r, r_1, s_1, r_2, s_2 \in \mathbb{Z}[\sqrt{-6}]$ we have:

(a) $\psi([r_1 2+s_1\sqrt{-6}]+[r_2 2+s_2\sqrt{-6}]) = \psi([r_1+r_2]2+[s_1+s_2]\sqrt{-6}) = (r_1+r_2)\alpha + (s_1+s_2)\beta = r_1\alpha + s_1\beta + r_2\alpha + s_2\beta = \psi(r_1 2 + s_1\sqrt{-6}) + \psi(r_2 2 + s_2\sqrt{-6})$,

(b) $\psi(r[r_1 2 + s_1\sqrt{-6}]) = \psi(rr_1 2 + rs_1\sqrt{-6}) = rr_1\alpha + rs_1\beta = r(r_1\alpha + s_1\beta) = r\psi(r_1 2 + s_1\sqrt{-6})$,

and finally we have that for every $r, s \in \mathbb{Z}[\sqrt{-6}]$:

$$g \circ \psi(r2 + s\sqrt{-6}) = g(r\alpha + s\beta) = rg(\alpha) + sg(\beta) = r2 + s\sqrt{-6} = \mathrm{id}_A(r2 + s\sqrt{-6})$$

hence $g \circ \psi = \mathrm{id}_A$ and ψ is a splitting, meaning that the exact sequence splits and thus A is projective.

Exercise 2

Let R be a ring with $1 \neq 0$ and M a finitely generated left R-module.

1. Suppose M is projective. We prove that there are elements $m_1, \ldots, m_k \in M$ and R-module homomorphisms $f_1, \ldots, f_k : M \longrightarrow R$ such that for all $m \in M$ we have $m = \sum_{i=1}^{k} f_i(m) m_i$. Since M is finitely generated, we know that there are elements $m_1, \ldots, m_k \in M$ (obviously the notation is intended) such that $M = \langle m_1, \cdots, m_k \rangle_R$. We now define a function from $Rm_1 + \cdots + Rm_k$, the free left R-module generated by those elements, to M by determining where we send the basis:
$$\begin{array}{rccccl} g & : & Rm_1 + \cdots + Rm_k & \longrightarrow & M & \\ & & m_i & \longmapsto & m_i & \end{array} \text{for } 1 \leq i \leq k.$$
Hence we have the exact sequence of left R-modules:
$$0 \to \ker(g) \xrightarrow{i} Rm_1 + \cdots + Rm_k \xrightarrow{g} M \to 0$$
which splits since M is projective. This means that there is a splitting, that is, a R-module homomorphism $\psi : M \longrightarrow Rm_1 + \cdots + Rm_k$ such that $g \circ \psi = \mathrm{id}_M$. Now, we have that for every $m \in M$ there are elements $r_i \in R$, $1 \leq i \leq k$ such that $m = \sum_{i=1}^{k} r_i m_i$ hence:
$$\psi(m) = \psi\left(\sum_{i=1}^{k} r_i m_i\right) = \sum_{i=1}^{k} \psi(r_i m_i) = \sum_{i=1}^{k} r_i \psi(m_i),$$
where we remark that the sums after the second equality are in $Rm_1 + \cdots + Rm_k$ and thus are only formal. We then define $f_i : M \longrightarrow R$ as $f_i(m) = r_i$ with the notation above. We have that f_i is a R-module homomorphism since for another $m' \in M$, say $\sum_{i=1}^{k} r'_i m_i$ for certain $r'_i \in R$, $1 \leq i \leq k$, and for an arbitrary $r \in R$ we have:
   (a) $\psi(m+m') = \psi(m)+\psi(m') = \sum_{i=1}^{k} r_i \psi(m_i) + \sum_{i=1}^{k} r'_i \psi(m_i) = \sum_{i=1}^{k} (r_i + r'_i)\psi(m_i)$ meaning that $f_i(m + m') = r_i + r'_i = f_i(m) + f_i(m')$,
   (b) $\psi(rm) = r\psi(m) = r \sum_{i=1}^{k} r_i \psi(m_i) = \sum_{i=1}^{k} r r_i \psi(m_i)$ meaning that $f_i(rm) = r r_i = r f_i(m)$.

   Moreover, we have that $m = \sum_{i=1}^{k} r_i m_i = \sum_{i=1}^{k} f_i(m) m_i$ by definition of f_i, hence we obtained what we desired.

2. Suppose that there are elements $m_1, \ldots, m_k \in M$ and R-module homomorphisms $f_1, \ldots, f_k : M \longrightarrow R$ such that for all $m \in M$ we have $m = \sum_{i=1}^{k} f_i(m) m_i$. We prove that M is projective. For this, let A, B be left R-modules and let:
$$0 \to A \to B \xrightarrow{g} M \to 0$$

be an exact sequence of left R-modules. We want to find a splitting, that is, a R-module homomorphism $\psi : M \longrightarrow B$ such that $g \circ \psi = \mathrm{id}_M$. Since g is surjective, there are elements $\alpha_1, \ldots, \alpha_k \in B$ with $g(\alpha_i) = m_i$ for $1 \le i \le k$. We thus define $\psi(m) = \sum_{i=1}^k f_i(m)\alpha_i$, which is clearly an element in B, for an arbitrary $m \in M$. We have that ψ is an R-module homomorphism since for another $m' \in M$, say $\sum_{i=1}^k r_i' m_i$ for certain $r_i' \in R$, $1 \le i \le k$, and for an arbitrary $r \in R$ we have:

(a) $\psi(m+m') = \sum_{i=1}^k f_i(m+m')\alpha_i = \sum_{i=1}^k (f_i(m) + f_i(m'))\alpha_i = \sum_{i=1}^k f_i(m)\alpha_i + \sum_{i=1}^k f_i(m')\alpha_i = \psi(m) + \psi(m')$,

(b) $\psi(rm) = \sum_{i=1}^k f_i(rm)\alpha_i = \sum_{i=1}^k r f_i(m)\alpha_i = r\sum_{i=1}^k f_i(m)\alpha_i = r\psi(m)$.

Finally, we check that indeed $g \circ \psi = \mathrm{id}_M$ since:

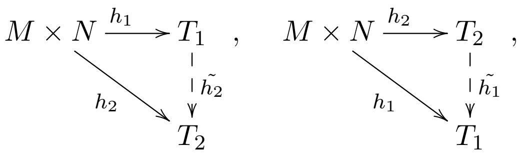
$$g \circ \psi(m) = g\left(\sum_{i=1}^k f_i(m)\alpha_i\right) = \sum_{i=1}^k g(f_i(m)\alpha_i) = \sum_{i=1}^k f_i(m)g(\alpha_i) = \sum_{i=1}^k f_i(m)m_i = m$$

and thus ψ is the desired splitting. Since this was done for an arbitrary exact sequence having M as the third R-module, we obtain that M is projective.

Exercise 3

Let M and N be right and left R-modules respectively for a ring R.

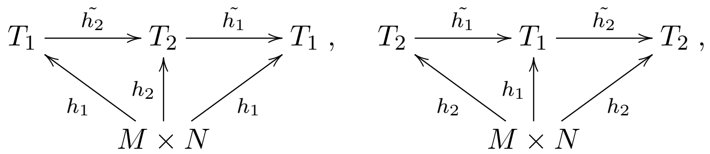
1. Show that $M \otimes_R N$ is unique up to unique isomorphism. For this, suppose we have (T_1, h_1) and (T_2, h_2) abelian groups satisfying the universal property of the tensor product. Applying the universal property respect to each other with their respective canonical R-biadditive maps, we obtain the commutative diagrams:



where $\tilde{h_1}$ and $\tilde{h_2}$ are unique with the respective properties:

$$\tilde{h_2} \circ h_1 = h_2$$
$$\tilde{h_1} \circ h_2 = h_1.$$

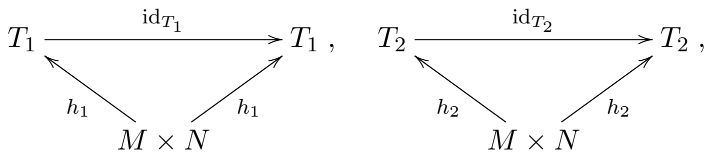
Hence we obtain the commutative diagrams:



if we are not convinced we can always verify that:

$$\tilde{h_1} \circ \tilde{h_2} \circ h_1 = \tilde{h_1} \circ h_2 = h_1$$
$$\tilde{h_2} \circ \tilde{h_1} \circ h_2 = \tilde{h_2} \circ h_1 = h_2.$$

However, note that the following diagrams are clearly commutative:



hence by the uniqueness of the morphisms that extend to the tensor product, we have that:

$$\tilde{h_1} \circ \tilde{h_2} = \mathrm{id}_{T_1}, \quad \tilde{h_2} \circ \tilde{h_1} = \mathrm{id}_{T_2}$$

thus $T_1 \cong T_2$ as abelian groups, and such an isomorphism is unique by the uniqueness of $\tilde{h_1}$ and $\tilde{h_2}$.

2. Suppose R is commutative and M, N are finitely generated as R-modules. Show that $M \otimes_R N$ is finitely generated as an R-module and determine a generating set. First, suppose $M = \langle \alpha_1, \dots, \alpha_m \rangle_R$ and $N = \langle \beta_1, \dots, \beta_n \rangle_R$. Then any pure tensor $m \otimes n \in M \otimes_R N$ can be written as:

$$m \otimes n = (r_1\alpha_1 + \cdots + r_m\alpha_m) \otimes (s_1\beta_1 + \cdots + s_n\beta_n) = \sum_{i,j} (r_i s_j)(\alpha_i \otimes \beta_j)$$

for certain $r_1, \dots, r_m, s_1, \dots, s_n \in R$ (we have heavily used that R is commutative to be able to multiply by the scalars outside the pure tensors). Now, since any element in $M \otimes_R N$ is a finite sum of pure tensors (maybe multiplied by some scalars in R), and the pure tensors are a (finite) sum of the form above, we obtain that the set $\{\alpha_i \otimes \beta_j : i = 1, \dots, m \text{ and } j = 1, \dots, n\}$ generates $M \otimes_R N$. Since $i = 1, \dots, m$ and $j = 1, \dots, n$ are finite, the generating set is finite and thus $M \otimes_R N$ is finitely generated.

Exercise 4

1. For $m, n \in \mathbb{N}^+$, show that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/t\mathbb{Z}$ for some $t \in \mathbb{N}^+$, and determine such t. We claim that $t = \gcd(m, n)$. To prove such an isomorphism, consider the map:

$$\begin{array}{rccc} \psi & : & \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Z}/t\mathbb{Z} \\ & & (\overline{a}, \overline{b}) & \longmapsto & \overline{ab} \end{array}.$$

First, note that ψ is well defined since if $(\overline{a}, \overline{b}) = (\overline{p}, \overline{q})$, that is $a = p + nk$ and $b = q + ms$ for some $k, s \in \mathbb{Z}$, then:

$$\psi(\overline{a}, \overline{b}) = \overline{ab} = \overline{(p + nk)(q + ms)} = \overline{pq} + \overline{pms} + \overline{qnk} + \overline{nkms} = \overline{pq} = \psi(\overline{p}, \overline{q})$$

since $\overline{m} = \overline{n} = 0$ in $\mathbb{Z}/t\mathbb{Z}$ because t divides m and n. Moreover, notice that ψ is $\mathbb{Z}$-biadditive since for $r, a, a_1, a_2, b, b_1, b_2 \in \mathbb{Z}$ we have:

   (a) $\psi(\overline{a_1} + \overline{a_2}, \overline{b}) = \psi(\overline{a_1 + a_2}, \overline{b}) = \overline{(a_1 + a_2)b} = \overline{a_1 b} + \overline{a_2 b} = \psi(\overline{a_1}, \overline{b}) + \psi(\overline{a_2}, \overline{b})$,

   (b) $\psi(\overline{a}, \overline{b_1} + \overline{b_2}) = \psi(\overline{a}, \overline{b_1 + b_2}) = \overline{a(b_1 + b_2)} = \overline{ab_1} + \overline{ab_2} = \psi(\overline{a}, \overline{b_1}) + \psi(\overline{a}, \overline{b_2})$,

   (c) $\psi(r\overline{a}, \overline{b}) = \psi(\overline{ra}, \overline{b}) = \overline{rab} = \overline{arb} = \psi(\overline{a}, \overline{rb}) = \psi(\overline{a}, r\overline{b})$ and $r\psi(\overline{a}, \overline{b}) = r\overline{ab} = \overline{rab}$, thus $r\psi(\overline{a}, \overline{b}) = \psi(r\overline{a}, \overline{b}) = \psi(\overline{a}, r\overline{b})$.

This means that ψ induces by the universal property of the tensor product a morphism of abelian groups $\phi : \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/t\mathbb{Z}$ such that $\phi(\overline{1} \otimes \overline{1}) = \psi(\overline{1}, \overline{1}) = \overline{1}$. Clearly $\overline{1} \otimes \overline{1}$ generates $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ since for $a, b \in \mathbb{N}$ any pure tensor $\overline{a} \otimes \overline{b} = (ab)(\overline{1} \otimes \overline{1})$. Now, notice that:

$$\begin{aligned} m(\overline{1} \otimes \overline{1}) &= \overline{m} \otimes \overline{1} = \overline{0} \otimes \overline{1} = 0, \\ n(\overline{1} \otimes \overline{1}) &= \overline{1} \otimes \overline{n} = \overline{1} \otimes \overline{0} = 0, \end{aligned}$$

hence the order of $\overline{1} \otimes \overline{1}$ in $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ divides m and n, thus it must divide $\gcd(m, n) = t$ (in particular since the order is positive, it is less than or equal to t). Moreover for any $k \in \mathbb{N}^+$ with $k < t$:

$$\phi(k(\overline{1} \otimes \overline{1})) = \phi(\overline{k} \otimes \overline{1}) = \phi(\overline{k}, \overline{1}) = \overline{k},$$

which is different than $\overline{0}$ in $\mathbb{Z}/t\mathbb{Z}$ because $k < t$. Hence since ϕ is an abelian group homomorphism, we must have $k(\overline{1} \otimes \overline{1}) \neq \overline{0}$ in $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ and thus the order of $\overline{1} \otimes \overline{1}$ must be greater or equal to t. Combining these two conditions we obtain that the order of $\overline{1} \otimes \overline{1}$ is exactly t, hence ϕ is an isomorphism of abelian groups.

2. We first show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as a $\mathbb{Q}$ vector space. For this, we define the map:

$$\begin{array}{rccc} \psi & : & \mathbb{Q} \times \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ & & \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) & \longmapsto & \frac{a_1 a_2}{b_1 b_2} \end{array}.$$

Notice that ψ is $\mathbb{Z}$-biadditive since for $p, q, a_1, a_2, b_1, b_2 \in \mathbb{Z}$ we have:

(a) $\psi(a_1/b_1{+}a_2/b_2, p/q) = \psi((a_1b_2{+}a_2b_1)/(b_1b_2), p/q) = (a_1b_2{+}a_2b_1)p/(b_1b_2q) = ((a_1b_2{+}a_2b_1)/(b_1b_2))(p/q) = (a_1/b_1{+}a_2/b_2)(p/q) = (a_1/b_1)(p/q){+}(a_2/b_2)(p/q) = a_1p/b_1q + a_2p/b_2q = \psi(a_1/b_1, p/q) + \psi(a_2/b_2, p/q)$,

(b) $\psi(p/q, a_1/b_1{+}a_2/b_2) = \psi(p/q, (a_1b_2{+}a_2b_1)/(b_1b_2)) = p(a_1b_2{+}a_2b_1)/(qb_1b_2) = (p/q)((a_1b_2{+}a_2b_1)/(b_1b_2)) = (p/q)(a_1/b_1{+}a_2/b_2) = (p/q)(a_1/b_1){+}(p/q)(a_2/b_2) = pa_1/qb_1 + pa_2/qb_2 = \psi(p/q, a_1/b_1) + \psi(p/q, a_2/b_2)$,

(c) $\psi(p(a_1/b_1), a_2/b_2) = \psi(pa_1/b_1, a_2/b_2) = pa_1a_2/b_1b_2 = a_1pa_2/b_1b_2 = \psi(a_1/b_1, pa_2/b_2) = \psi(a_1/b_1, p(a_2/b_2))$ and $p\psi(a_1/b_1, a_2/b_2) = p(a_1a_2/b_1b_2) = pa_1a_2/b_1b_2$, thus $p\psi(a_1/b_1, a_2/b_2) = \psi(p(a_1/b_1), a_2/b_2) = \psi(a_1/b_1, p(a_2/b_2))$.

This means that ψ induces by the universal property of the tensor product a morphism of abelian groups $\phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$ such that $\phi(1 \otimes 1) = \psi(1, 1) = \overline{1}$. We first want to define a multiplication by scalars on $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, which we do by means of the pure tensors as:

$$\begin{array}{ccc} \mathbb{Q} \times \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \\ \left(\frac{p}{q}, \frac{a_1}{b_1} \otimes \frac{a_2}{b_2}\right) & \longmapsto & \frac{pa_1}{qb_1} \otimes \frac{a_2}{b_2} \end{array}$$

and we extend to $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ by linearity. The axioms of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ being a $\mathbb{Q}$ module with the multiplication above follow from the linearity on the second component (requested by definition) and the properties of the multiplication in $\mathbb{Q}$.

Notice how for $p, q, a_1, a_2, b_1, b_2 \in \mathbb{Z}$ we have:

$$\frac{pa_1}{qb_1} \otimes \frac{a_2}{b_2} = \frac{a_1}{qb_1} \otimes \frac{pa_2}{b_2} = \frac{a_1}{qb_1} \otimes \frac{qpa_2}{qb_2} = \frac{qa_1}{qb_1} \otimes \frac{pa_2}{qb_2} = \frac{a_1}{b_1} \otimes \frac{pa_2}{qb_2}.$$

This means that as a $\mathbb{Q}$ module, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by $1 \otimes 1$ since for $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ we have $(a_1/b_1) \otimes (a_2/b_2) = (a_1a_2/b_1b_2) \otimes 1 = (a_1a_2/b_1b_2)(1 \otimes 1)$, and hence for finite sums of pure tensors we can always reduce to $1 \otimes 1$ multiplied by a scalar in $\mathbb{Q}$ (in particular every element in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a pure tensor). Now, we can prove that ϕ is a linear map that is a bijection (we can reduce us by the above to the case where we only deal with pure tensors):

(a) ϕ is $\mathbb{Q}$ linear since it is already a group homomorphism by the universal property of the tensor product that defined it, and for $a, b, a_1, a_2, b_1, b_2 \in \mathbb{Z}$ we have:

$$\begin{aligned} \phi\left(\left(\frac{a}{b}\right)\left(\frac{a_1}{b_1} \otimes \frac{a_2}{b_2}\right)\right) &= \phi\left(\frac{aa_1}{bb_1} \otimes \frac{a_2}{b_2}\right) = \psi\left(\frac{aa_1}{bb_1}, \frac{a_2}{b_2}\right) = \frac{aa_1a_2}{bb_1b_2} \\ &= \frac{a}{b}\frac{a_1a_2}{b_1b_2} = \left(\frac{a}{b}\right)\psi\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = \left(\frac{a}{b}\right)\phi\left(\frac{a_1}{b_1} \otimes \frac{a_2}{b_2}\right). \end{aligned}$$

(b) ϕ is injective: for a_1, a_2, b_1, b_2 we have $\phi((a_1/b_1) \otimes (a_2/b_2)) = 0$, this means $(a_1a_2)/(b_1b_2) = 0$ hence $a_1a_2 = 0$ and thus $a_1 = 0$ or $a_2 = 0$, meaning that $a_1/b_1 = 0$ or $a_2/b_2 = 0$, in either case $(a_1/b_1) \otimes (a_2/b_2) = 0$, thus the kernel is trivial.

(c) ϕ is surjective: for $a/b \in \mathbb{Q}$ we have $\phi((a/b) \otimes 1) = \psi(a/b, 1) = a/b$.

Hence indeed ϕ is an isomorphism of $\mathbb{Q}$ vector spaces and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

We now prove that $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} \not\cong \mathbb{R}$. In an analogous way as we proved above, we have that $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ is an $\mathbb{R}$ vector space via:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} & \longrightarrow & \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} \\ (r, r_1 \otimes r_2) & \longmapsto & (rr_1) \otimes r_2 \end{array}$$

and we extend to $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ by linearity. We know that $\mathbb{R}$ is an infinite dimensional $\mathbb{Q}$ vector space, consider $\{\alpha_i\}_{i \in I}$ a $\mathbb{Q}$ basis for R. We claim that $\{1 \otimes \alpha_i\}_{i \in I}$ is an $\mathbb{R}$ basis for $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$. First, it generates the pure tensors (hence it generates the whole space) since for $r, s \in \mathbb{R}$ we have:

$$r \otimes s = r \otimes \left(\sum_{i \in I} q_i \alpha_i \right) = \sum_{i \in I} r \otimes (q_i \alpha_i) = \sum_{i \in I} (q_i r) \otimes \alpha_i = \sum_{i \in I} (r q_i)(1 \otimes \alpha_i)$$

for some $q_i \in \mathbb{Q}$ for $i \in I$ (all zero but a finite number, thus the sum is finite). Second, to see that they are linearly independent, we are interested in the isomorphisms $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R} \otimes_{\mathbb{Q}} \left(\bigoplus_{i \in I} \mathbb{Q}\alpha_i \right) \cong \bigoplus_{i \in I} (\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}\alpha_i)$, whose composition sends (in the notation above) a pure tensor $r \otimes s$ to its coordinates $(rq_i)_{i \in I}$. Thus when we have that a (finite) linear combination of $\{1 \otimes \alpha_i\}_{i \in I}$ is zero in $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$, via the isomorphisms we have that in the direct sum all the coordinates are zero, hence all the coefficients that multiplied $\{1 \otimes \alpha_i\}_{i \in I}$ must have been zero already.

Thus, we found that $\dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}) = \dim_{\mathbb{Q}}(\mathbb{R}) > 1$, but $\dim_{\mathbb{R}}(\mathbb{R}) = 1$, thus we cannot have that $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{R}$ are isomorphic because their bases have different cardinality, hence $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} \not\cong \mathbb{R}$ as desired.

Exercise 5

Let R be commutative, M, N, P be R-modules. We want to show that $\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. For this, we will build two morphisms of R-modules $\psi : \mathrm{Hom}_R(M \otimes_R N, P) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ and $\phi : \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \longrightarrow \mathrm{Hom}_R(M \otimes_R N, P)$ such that $\phi \circ \psi = \mathrm{id}_{\mathrm{Hom}_R(M \otimes_R N, P)}$ and $\psi \circ \phi = \mathrm{id}_{\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))}$.

We start by letting $f \in \mathrm{Hom}_R(M \otimes_R N, P)$, say:

$$\begin{array}{rcccc} f & : & M \otimes_R N & \longrightarrow & P \\ & & m \otimes n & \longmapsto & f(m \otimes n) \end{array}$$

then we define:

$$\begin{array}{rcccc} \psi & : & \mathrm{Hom}_R(M \otimes_R N, P) & \longrightarrow & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \\ & & f & \longmapsto & \psi(f) \end{array}$$

by setting for $m \in M$ the function $\psi(f)(m) : N \longrightarrow P$ as:

$$\begin{array}{rcccc} \psi(f)(m) & : & N & \longrightarrow & P \\ & & n & \longmapsto & f(m \otimes n) \end{array}$$

so that $\psi(f)(m)(n) = f(m \otimes n)$. To verify that this is well defined, we need that $\psi(f)(m) \in \mathrm{Hom}_R(N, P)$ and $\psi(f) \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. Let $n, n_1, n_2 \in N$ and $r \in R$, we have:

1. $\psi(f)(m)(n_1+n_2) = f(m\otimes(n_1+n_2)) = f(m\otimes n_1+m\otimes n_2) = f(m\otimes n_1)+f(m\otimes n_2) = \psi(f)(m)(n_1) + \psi(f)(m)(n_2)$,
2. $\psi(f)(m)(rn) = f(m \otimes (rn)) = f(r(m \otimes n)) = rf(m \otimes n) = r\psi(f)(m)(n)$,

where we heavily used that f is a morphism of R-modules, hence $\psi(f)(m) \in \mathrm{Hom}_R(N, P)$. Let $m_1, m_2 \in M$, we have:

1. $\psi(f)(m_1 + m_2)(n) = f((m_1 + m_2) \otimes n) = f(m_1 \otimes n + m_2 \otimes n) = f(m_1 \otimes n) + f(m_2 \otimes n) = \psi(f)(m_1)(n) + \psi(f)(m_2)(n)$,
2. $\psi(f)(rm)(n) = f((rm) \otimes n) = f(r(m \otimes n)) = rf(m \otimes n) = r\psi(f)(m)(n)$,

where we again used that f is a morphism of R-modules, hence $\psi(f) \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. Thus, ψ is well defined. We now prove that ψ is an R-module morphism, let $f, f_1, f_2 \in \mathrm{Hom}_R(M \otimes_R N, P)$, we have:

1. $\psi(f_1 + f_2)(m)(n) = (f_1 + f_2)(m \otimes n) = f_1(m \otimes n) + f_2(m \otimes n) = \psi(f_1)(m)(n) + \psi(f_2)(m)(n)$,
2. $\psi(rf)(m)(n) = (rf)(m \otimes n) = f(r(m \otimes n)) = rf(m \otimes n) = r\psi(f)(m)(n)$,

where we rely on f being a morphism of R-modules, hence ψ is as desired.

Now, we continue by letting $\alpha \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$, say:

$$\begin{array}{cccc} \alpha & : & M & \longrightarrow & \mathrm{Hom}_R(N,P) \\ & & m & \longmapsto & \alpha(m) \end{array}$$

then we define:

$$\begin{array}{cccc} \tilde{\alpha} & : & M \times N & \longrightarrow & P \\ & & (m,n) & \longmapsto & \alpha(m)(n) \end{array}$$

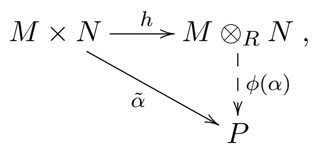
which is R-biadditive since for $r \in R$, $m, m' \in M$, $n, n' \in N$ we have:

1. $\tilde{\alpha}(m+m', n) = \alpha(m+m')(n) = (\alpha(m)+\alpha(m'))(n) = \alpha(m)(n) + \alpha(m')(n) = \tilde{\alpha}(m,n) + \tilde{\alpha}(m',n)$ since $\alpha \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$,

2. $\tilde{\alpha}(m, n+n') = \alpha(m)(n+n') = \alpha(m)(n) + \alpha(m)(n') = \tilde{\alpha}(m,n) + \tilde{\alpha}(m,n')$ since $\alpha(m) \in \mathrm{Hom}_R(N,P)$,

3. $\tilde{\alpha}(rm, n) = \alpha(rm)(n) = r\alpha(m)(n) = r\tilde{\alpha}(m,n)$ since $\alpha \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$ and $\tilde{\alpha}(m, rn) = \alpha(m)(rn) = r\alpha(m)(n) = r\tilde{\alpha}(m,n)$ since $\alpha(m) \in \mathrm{Hom}_R(N,P)$, hence $\tilde{\alpha}(rm,n) = r\tilde{\alpha}(m,n) = \tilde{\alpha}(m,rn)$.

This means that by the universal property of the tensor product, there is a unique group homomorphism, that we name $\phi(\alpha)$, such that the following diagram commutes:



that is $\tilde{\alpha} = \phi(\alpha) \circ h$, otherwise said, over the pure tensors $m \otimes n \in M \otimes_R N$ we have $\phi(\alpha)(m \otimes n) = \tilde{\alpha}(m,n) = \alpha(m)(n)$. Then we define:

$$\begin{array}{cccc} \phi & : & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P)) & \longrightarrow & \mathrm{Hom}_R(M \otimes_R N, P) \\ & & \alpha & \longmapsto & \phi(\alpha) \end{array}$$

by setting $\phi(\alpha)$ as above. To verify that this is well defined, we need that $\phi(\alpha) \in \mathrm{Hom}_R(M \otimes_R N, P)$. Let $m \otimes n \in M \otimes_R N$, we have:

1. Since $\phi(\alpha)$ is a group homomorphism, we already know that behaves as desired when applied to a sum of pure tensors.

2. $\phi(\alpha)(r(m \otimes n)) = \phi(\alpha)((rm) \otimes n) = \alpha(rm)(n) = r\alpha(m)(n) = r\phi(\alpha)(m \otimes n)$,

where we used that $\alpha \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$, hence $\phi(\alpha) \in \mathrm{Hom}_R(M \otimes_R N, P)$ and thus ϕ is well defined. We now prove that ϕ is an R-module morphism, let $\alpha, \alpha_1, \alpha_2 \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$, we have:

1. $\phi(\alpha_1+\alpha_2)(m\otimes n)=(\alpha_1+\alpha_2)(m)(n)=\alpha_1(m)(n)+\alpha_2(m)(n)=\phi(\alpha_1)(m\otimes n)+\phi(\alpha_2)(m\otimes n)$,
2. $\phi(r\alpha)(m\otimes n)=(r\alpha)(m)(n)=\alpha(rm)(n)=r\alpha(m)(n)=r\phi(\alpha)(m\otimes n)$,

where we rely on α being a morphism of R-modules, hence ϕ is as desired.

Finally, we have that:

$$\begin{aligned}\phi\circ\psi(f)(m\otimes n) &= \psi(f)(m)(n)=f(m\otimes n)=\mathrm{id}_{\mathrm{Hom}_R(M\otimes_R N,P)}(f)(m\otimes n)\\ \psi\circ\phi(\alpha)(m)(n) &= \phi(\alpha)(m\otimes n)=\alpha(m)(n)=\mathrm{id}_{\mathrm{Hom}_R(M,\mathrm{Hom}_R(N,P))}(\alpha)(m)(n)\end{aligned}$$

hence indeed $\phi\circ\psi=\mathrm{id}_{\mathrm{Hom}_R(M\otimes_R N,P)}$ and $\psi\circ\phi=\mathrm{id}_{\mathrm{Hom}_R(M,\mathrm{Hom}_R(N,P))}$, and $\mathrm{Hom}_R(M\otimes_R N,P)\cong\mathrm{Hom}_R(M,\mathrm{Hom}_R(N,P))$.