Algebra II - Homework 5

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Exercise 1

Let R and S be rings, A a right R-module, B an (R,S)-bimodule, C a left S-module. We want to prove that $A \otimes_R (B \otimes_S C) \cong (A \otimes_R B) \otimes_S C$ as abelian groups.

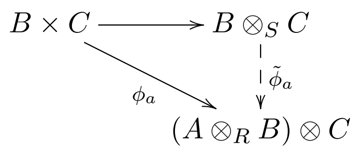
First, we want to define a group homomorphism $\phi : A \otimes_R (B \otimes_S C) \longrightarrow (A \otimes_R B) \otimes_S C$. For this, for every element $a \in A$ we consider the maps:

$$\begin{array}{rccc} \phi_a & : & B \times C & \longrightarrow & (A \otimes_R B) \otimes_S C \\ & & (b,c) & \longrightarrow & (a \otimes b) \otimes c \end{array}$$

which are S-biadditive, since given $b, b' \in B$, $c, c' \in C$ and $s \in S$ we have:

1. $\phi_a(b+b',c) = (a \otimes (b+b')) \otimes c = (a \otimes b + a \otimes b') \otimes c = (a \otimes b) \otimes c + (a \otimes b') \otimes c = \phi_a(b,c) + \phi_a(b',c)$,
2. $\phi_a(b, c+c') = (a \otimes b) \otimes (c+c') = (a \otimes b) \otimes c + (a \otimes b) \otimes c' = \phi_a(b,c) + \phi_a(b,c')$,
3. $\phi_a(bs, c) = (a \otimes (bs)) \otimes c = ((a \otimes b)s) \otimes c = (a \otimes b) \otimes (sc) = \phi_a(b, sc)$.

This yields the commutative diagram:



that sets $\tilde{\phi}_a(b \otimes c) = (a \otimes b) \otimes c$ a group homomorphism. Now, we consider the map:

$$\begin{array}{rccc} \overline{\phi} & : & A \times (B \otimes_S C) & \longrightarrow & (A \otimes_R B) \otimes_S C \\ & & (a, b \otimes c) & \longrightarrow & \tilde{\phi}_a(b \otimes c) \end{array}$$

which is well defined since if we have $b, b' \in B$, $c, c' \in C$ with $b \otimes c = b' \otimes c'$ then for any $a \in A$ we have $\overline{\phi}(a, b \otimes c) = \tilde{\phi}_a(b \otimes c) = \tilde{\phi}_a(b' \otimes c') = \overline{\phi}(a, b' \otimes c')$, where we have used that $\tilde{\phi}_a$ is well defined on $B \otimes_S C$ by construction. Now $\overline{\phi}$ is R-biadditive, since given $a, a' \in A$, $b, b' \in B$, $c, c' \in C$ and $r \in R$ we have:

1. $\overline{\phi}(a+a', b \otimes c) = ((a+a') \otimes b) \otimes c = (a \otimes b + a' \otimes b) \otimes c = (a \otimes b) \otimes c + (a' \otimes b) \otimes c = \overline{\phi}(a, b \otimes c) + \overline{\phi}(a', b \otimes c)$,
2. $\overline{\phi}(a, b \otimes c + b' \otimes c') = \tilde{\phi}_a(b \otimes c + b' \otimes c') = \tilde{\phi}_a(b \otimes c) + \tilde{\phi}_a(b' \otimes c') = \overline{\phi}(a, b \otimes c) + \overline{\phi}(a, b' \otimes c')$ since $\tilde{\phi}_a$ is a group homomorphism,
3. $\overline{\phi}(ar, b \otimes c) = ((ar) \otimes b) \otimes c = (a \otimes (rb)) \otimes c = \overline{\phi}(a, (rb) \otimes c) = \overline{\phi}(a, r(b \otimes c))$,

where we remark that it is enough to prove the properties for pure tensors since for a general sum of pure tensors we just apply the fact that $\tilde{\phi}_a$ is a group homomorphism to divide the sum into the pure tensors, use the properties above and then regroup the terms as needed to obtain the result. This yields the commutative diagram:

$$\begin{array}{ccc} A \times (B \otimes_S C) & \longrightarrow & A \otimes_R (B \otimes_S C) \\ & \overline{\phi} \searrow & \Big\downarrow \phi \\ & & (A \otimes_R B) \otimes C \end{array}$$

that sets $\phi(a \otimes (b \otimes c)) = (a \otimes b) \otimes c$ a group homomorphism.

Second, we proceed in an perfectly analogous way in the other direction. We omit the details to not appear ridiculously redundant, so first working on $\psi_c : A \times B \longrightarrow A \otimes_R (B \otimes_S C)$ in the exact same way and then on $\overline{\psi} : (A \otimes_R B) \times C \longrightarrow A \otimes_R (B \otimes_S C)$ in again the exact same way, we obtain a group homomorphism $\psi : (A \otimes_R B) \otimes_S C \longrightarrow A \otimes_R (B \otimes_S C)$ that sets $\psi((a \otimes b) \otimes c) = a \otimes (b \otimes c)$.

Finally, it is easy to check that for $a \in A$, $b \in B$, $c \in C$ we have:

$$\psi \circ \phi(a \otimes (b \otimes c)) = \psi((a \otimes b) \otimes c) = a \otimes (b \otimes c)$$
$$\phi \circ \psi((a \otimes b) \otimes c) = \phi(a \otimes (b \otimes c)) = a \otimes (b \otimes c)$$

hence indeed $\psi \circ \phi = \mathrm{id}_{A \otimes_R (B \otimes_S B)}$ and $\phi \circ \psi = \mathrm{id}_{(A \otimes_R B) \otimes_S B}$ so ϕ is a group isomorphism with inverse ψ. Hence indeed $A \otimes_R (B \otimes_S C) \cong (A \otimes_R B) \otimes_S C$ as abelian groups, as desired.

Exercise 2

Let R be a commutative ring.

1. Let F be a flat R-module and $a \in R$ a non-zero divisor. Show that if $ax = 0$ for some $x \in F$ then $x = 0$. For this, consider the map $\phi : R \longrightarrow R$ given by $\phi(r) = ra$ for every $r \in R$. Since a is a non-zero divisor, if we have two elements $s, t \in R$ with $as = at$, this means $a(s-t) = 0$ thus $s = t$, and ϕ is injective. Now, consider:

$$0 \to R \xrightarrow{\phi} R \xrightarrow{\pi} Y/\text{coker}(\phi) \to 0$$

with π being the natural projection. This sequence is exact by the fact that ϕ is injective and the definition of cokernel. Since F is flat, when we tensor by F via R we obtain the exact sequence:

$$0 \to F \otimes_R R \xrightarrow{1\otimes\phi} F \otimes_R R \xrightarrow{1\otimes\pi} F \otimes_R Y/\text{coker}(\phi) \to 0.$$

Hence $1 \otimes \phi$ is an injective function, and we have:

$$(1 \otimes \phi)(x \otimes 1) = x \otimes a = xa \otimes 1 = 0 \otimes 1 = 0 \otimes a = (1 \otimes \phi)(0 \otimes 1)$$

which means that $x \otimes 1 = 0 \otimes 1$ by injectivity. Recalling that $F \otimes_R R$ is isomorphic to F as R-modules via the map $\psi : F \otimes_R R \longrightarrow F$ given by $\psi(f \otimes r) = fr$ for $f \in F$ and $r \in R$, we have that $x = \phi(x \otimes 1) = \phi(0 \otimes 1) = 0$, as desired.

2. Let R be a principal ideal domain. We prove that a finitely generated R-module M is flat if and only if it is torsion free.

$\Rightarrow$) Let M be flat, suppose there is an element $x \in M_{tors}$, that is, there is a non-zero element $a \in R$ such that $ax = 0$. Since R is a domain, a is not a zero divisor. Hence by the section above $x = 0$ and $M_{tors} = \{0\}$, that is, M is torsion free.

$\Leftarrow$) Let M be torsion free and finitely generated. Over R a principal ideal domain, we have proven in class that this implies that M is free. This means that M is projective as R module, which in turn means that M is flat.

Exercise 3

Let $d \in \mathbb{Z}$ not a perfect square, consider $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}$. Let $p, q \in \mathbb{Z}$ be primes.

1. We want to define a surjective $\mathbb{Q}$-algebra homomorphism $\phi : \mathbb{Q}(\sqrt{p})[x] \longrightarrow \mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q})$ such that $\ker(\phi) = (x^2 - q)$. For this, we define $\phi(1) = 1 \otimes 1$, $\phi(\sqrt{p}) = \sqrt{p} \otimes 1$ and $\phi(x) = 1 \otimes \sqrt{q}$ and then we extend by linearity. We notice that since we are defining a function from a polynomial ring over a field, it suffices to define the image of the generator(s) of the ring, which in our case are the variable and the two generators of the field, hence ϕ is well defined.

   We now check that ϕ is a $\mathbb{Q}$-algebra homomorphism. Let $f, g \in \mathbb{Q}(\sqrt{p})[x]$, adding zero coefficients if necessary we can assume that f and g have the same degree, say $f = \sum_{i=1}^n f_i x^i$, $g = \sum_{i=1}^n g_i x^i$ with $f_i, g_i \in \mathbb{Q}(\sqrt{p})$ for $i = 1, \ldots, n$. Let $r \in \mathbb{Q}(\sqrt{p})$.

   (a) We have:

$$\begin{aligned} \phi(f+g) &= \phi\left(\sum_{i=1}^n (f_i + g_i) x^i\right) = \sum_{i=1}^n \phi(f_i + g_i)\phi(x)^i \\ &= \sum_{i=1}^n (\phi(f_i) + \phi(g_i))\phi(x)^i = \sum_{i=1}^n \phi(f_i)\phi(x)^i + \sum_{i=1}^n \phi(g_i)\phi(x)^i \\ &= \phi(f) + \phi(g), \end{aligned}$$

   where we have used that $\phi(f_i + g_i) = \phi(f_i) + \phi(g_i)$. To check this, it is enough to prove that $\phi(a+b) = \phi(a) + \phi(b)$ for $a, b \in \mathbb{Q}(\sqrt{p})$, say say $a = a_1 + a_2\sqrt{p}$, $b = b_1 + b_2\sqrt{p}$ with $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. Now:

$$\begin{aligned} \phi(a+b) &= \phi(a_1 + b_1 + (a_2 + b_2)\sqrt{p}) = (a_1 + b_1)\phi(1) + (a_2 + b_2)\phi(\sqrt{p}) \\ &= (a_1 + b_1) 1 \otimes 1 + (a_2 + b_2)\sqrt{p} \otimes 1 = a_1 1 \otimes 1 + b_1 1 \otimes 1 \\ &+ a_2\sqrt{p} \otimes 1 + b_2\sqrt{p} \otimes 1 = a_1\phi(1) + a_2\phi(\sqrt{p}) + b_1\phi(1) + b_2\phi(\sqrt{p}) \\ &= \phi(a_1 + a_2\sqrt{p}) + \phi(b_1 + b_2\sqrt{p}) = \phi(a) + \phi(b). \end{aligned}$$

   (b) We have:

$$\begin{aligned} \phi(fg) &= \phi\left(\sum_{i,j} f_i g_j x^{i+j}\right) = \sum_{i,j} \phi(f_i g_j)\phi(x)^{i+j} = \sum_{i,j} \phi(f_i)\phi(g_j)\phi(x)^i\phi(x)^j \\ &= \left(\sum_{i=1}^n \phi(f_i)\phi(x)^i\right)\left(\sum_{i=1}^n \phi(g_i)\phi(x)^i\right) = \phi(f)\phi(g) \end{aligned}$$

   where we have used that $\phi(f_i g_j) = \phi(f_i)\phi(g_j)$. To check this, it is enough to prove that $\phi(ab) = \phi(a)\phi(b)$ for $a, b \in \mathbb{Q}(\sqrt{p})$, say say $a = a_1 + a_2\sqrt{p}$,

$b = b_1 + b_2\sqrt{p}$ with some $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. Now:

$$\begin{aligned}
\phi(ab) &= \phi(a_1b_1 + (a_1b_2 + a_2b_1)\sqrt{p} + a_2b_2p) = (a_1b_1 + a_2b_2p)\phi(1) \\
&+ (a_1b_2 + a_2b_1)\phi(\sqrt{p}) = (a_1b_1 + a_2b_2p)1 \otimes 1 + (a_1b_2 \\
&+ a_2b_1)\sqrt{p} \otimes 1 = a_1b_1 1 \otimes 1 + a_2b_2p \otimes 1 + (a_1b_2 + a_2b_1)\sqrt{p} \otimes 1 \\
&= (a_1 1 \otimes 1 + a_2\sqrt{p} \otimes 1)(b_1 1 \otimes 1 + b_2\sqrt{p} \otimes 1) = (a_1\phi(1) \\
&+ a_2\phi(\sqrt{p}))(b_1\phi(1) + b_2\phi(\sqrt{p})) = \phi(a)\phi(b),
\end{aligned}$$

where we have used the multilinearity of the tensor product.

(c) By definition $\phi(1) = 1 \otimes 1$.

(d) We have:

$$\begin{aligned}
\phi(rf) &= \phi\left(\sum_{i=1}^{n} rf_i x^i\right) = \sum_{i=1}^{n} \phi(rf_i)\phi(x)^i = \sum_{i=1}^{n} r\phi(f_i)\phi(x)^i \\
&= r\sum_{i=1}^{n} \phi(f_i)\phi(x)^i = r\phi(f),
\end{aligned}$$

where we have used that $\phi(rf_i) = r\phi(f_i)$. To check this, it is enough to prove that $\phi(ra) = r\phi(a)$ for $a, \in \mathbb{Q}(\sqrt{p})$, say say $a = a_1 + a_2\sqrt{p}$ with some $a_1, a_2 \in \mathbb{Q}$. Now:

$$\begin{aligned}
\phi(ra) &= \phi(ra_1 + ra_2\sqrt{p}) = ra_1\phi(1) + ra_2\phi(\sqrt{p}) = r(a_1\phi(1) + a_2\phi(\sqrt{p})) \\
&= r\phi(a_1 + a_2\sqrt{p}) = r\phi(a).
\end{aligned}$$

This proves that ϕ is a $\mathbb{Q}$-algebra homomorphism.

Clearly ϕ is surjective since for any pure tensor $s \otimes t \in \mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q})$ we have $s = s_1 + s_2\sqrt{p}$ and $t = t_1 + t_2\sqrt{q}$ with some $s_1, s_2, t_1, t_2 \in \mathbb{Q}$. Then:

$$\begin{aligned}
\phi(s_1t_1 &+ s_2t_1\sqrt{p} + s_1t_2x + s_2t_2\sqrt{p}x) = (s_1t_1)1 \otimes 1 + (s_2t_1)\sqrt{p} \otimes 1 \\
&+ (s_1t_2)1 \otimes \sqrt{q} + (s_2t_2)\sqrt{p} \otimes \sqrt{q} = s_1 \otimes t_1 + s_2\sqrt{p} \otimes t_1 \\
&+ s_1 \otimes t_2\sqrt{q} + s_2\sqrt{p} \otimes t_2\sqrt{q} = (s_1 + s_2\sqrt{p}) \otimes (t_1 + t_2\sqrt{q}) = s \otimes t,
\end{aligned}$$

where the multilinearity of the tensor product has been used. If any pure tensor belongs to the image, by the linearity of ϕ any sum of pure tensors belongs to the image.

We just have to prove that $\ker(\phi) = (x^2 - q)$.

$\supseteq$) Clearly $\phi(x^2 - q) = \phi(x)^2 - q\phi(1) = 1 \otimes q - q \otimes 1 = 0$.

$\supseteq$ Let $f \in \ker(\phi)$, since $\mathbb{Q}(\sqrt{p})$ is a field, we have that $\mathbb{Q}(\sqrt{p})[x]$ is an Euclidean domain. In fact, it suffices that it has a division algorithm. We divide by $x^2 - q$ to obtain a decomposition $f = g(x^2 - q) + h$ where $g, h \in \mathbb{Q}(\sqrt{p})[x]$ and $\deg(h) < 2$.

This means that $h = h_0 + h_1 x$ with $h_0, h_1 \in \mathbb{Q}(\sqrt{p})$, say $h_0 = h_{0,1} + h_{0,2}\sqrt{p}$ and $h_1 = h_{1,1} + h_{2,2}\sqrt{q}$ with some $h_{0,1}, h_{0,2}, h_{1,1}, h_{1,2} \in \mathbb{Q}$. Now applying ϕ we obtain:

$$\begin{aligned} 0 &= \phi(f) = \phi(g(x^2 - q) + h) = \phi(g)\phi(x^2 - q) + \phi(h) = \phi(h) = h_{0,1}\phi(1) \\ &+ h_{0,2}\phi(\sqrt{p}) + h_{1,1}\phi(x) + h_{1,2}\phi(\sqrt{p})\phi(x) = h_{0,1} 1 \otimes 1 + h_{0,2}\sqrt{p} \otimes 1 \\ &+ h_{1,1} 1 \otimes \sqrt{q} + h_{1,2}\sqrt{p} \otimes \sqrt{q}, \end{aligned}$$

that is, a $\mathbb{Q}$-linear combination of the pure tensors $1 \otimes 1, \sqrt{p} \otimes 1, 1 \otimes \sqrt{q}, \sqrt{p} \otimes \sqrt{q}$. However, we note that since they always differ in a factor of $\sqrt{p}$ or $\sqrt{q}$ inside at least one component of tensor product, and those factors cannot be multilinearly transferred to the other component since they do not belong in $\mathbb{Q}$, we have that those four pure tensors are $\mathbb{Q}$-linearly independent. Thus we must have $h_{0,1} = h_{0,2} = h_{1,1} = h_{1,2} = 0$, meaning that $h = 0$ and $f \in (x^2 - q)$.

We have thus constructed a $\mathbb{Q}$-algebra homomorphism that is surjective with kernel $(x^2 - q)$, as desired.

2. If $p \neq q$, show that $\mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q})$ is a field. Applying the First Isomorphism Theorem to the section above, we obtain that $\mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q}) \cong \mathbb{Q}(\sqrt{p})[x]/(x^2 - q)$. Since everything is commutative, for this quotient to be a field it is enough that the ideal $(x^2 - q)$ is maximal, which happens if and only if the polynomial $x^2 - q \in \mathbb{Q}(\sqrt{p})[x]$ is irreducible. Note that $x^2 - q$ has no roots, since the solutions are $\pm\sqrt{q} \notin \mathbb{Q}(\sqrt{p})$ since $p \neq q$ and q is not a perfect square. Since $x - q$ is of degree two, this means that it cannot decompose in factors of degree one, meaning that it is indeed irreducible, as desired. Hence $\mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q})$ is a field.

3. Show that $\mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{p})$ is not a domain by finding a zero divisor. For this, we consider $a, b, f, g \in \mathbb{Q}$ and impose the multiplication:

$$\begin{aligned} 0 &= (a \otimes p + b \otimes \sqrt{p})(f \otimes p + g \otimes \sqrt{p}) \\ &= af \otimes p^2 + ag \otimes p\sqrt{p} + bf \otimes p\sqrt{p} + bg \otimes p \\ 0 &= afp \otimes 1 + ag \otimes \sqrt{p} + bf \otimes \sqrt{p} + bg \otimes 1 \end{aligned}$$

where we have divided by p to obtain the second equation. Factoring out and imposing that the first components in the tensor product are zero, we obtain the system of equations:

$$\begin{cases} afp + bg = 0 \\ ag + bf = 0 \end{cases} \implies \quad b = a\sqrt{p}, \quad g = -f\sqrt{p}$$

which is a solution of the first equation, and also works for the second. Now choosing $a = 1 = f$, we obtain that:

$$(1 \otimes p + (-1) \otimes \sqrt{p})(1 \otimes p + 1 \otimes \sqrt{p}) = 1 \otimes p^2 + 1 \otimes p\sqrt{p} + (-1) \otimes p\sqrt{p} + (-1) \otimes p = 0$$

thus $1 \otimes p + 1 \otimes \sqrt{p}$ is a zero divisor and $\mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{p})$ is not a domain, as desired.