Algebra II - Homework 6

Pablo Sánchez Ocal

April 5th, 2017

Exercise 1

We have R a commutative ring (with $1 \neq 0$), M and N are R-modules such that M is finitely generated and N is Noetherian. We show that $M \otimes_R N$ is Noetherian.

Suppose we have $M = \langle m_1, \dots, m_k \rangle_R$, we have seen multiple times in class that this means that the map $\phi : Rm_1 \oplus \cdots \oplus Rm_k \longrightarrow M$ given by $\phi(m_i) = m_i$ for $i = 1, \dots, k$ is a surjective R-module homomorphism. Thus we have the exact sequence of R-modules:

$$0 \longrightarrow \ker(\phi) \xrightarrow{\iota} R^k \xrightarrow{\phi} M \longrightarrow 0,$$

where ι denotes the natural inclusion. Applying the right exact functor $\cdot \otimes_R N$, we obtain the exact sequence:

$$0 \longrightarrow \ker(\phi \otimes \mathrm{id}_N) \longrightarrow R^k \otimes_R N \xrightarrow{\phi \otimes \mathrm{id}_N} M \otimes_R N \longrightarrow 0,$$

where the first morphism is the natural inclusion: note that the right exactness only guarantees that the morphism $\phi \otimes \mathrm{id_N}$ is surjective, but we can always make a surjective morphism into a short exact sequence by including its kernel before it. Now, since $R \otimes_R N \cong N$ as R-modules, we have that $R^k \otimes_R N \cong N^k$ and the exact sequence of R-modules:

$$0 \longrightarrow \ker(\phi \otimes \mathrm{id}_N) \longrightarrow N^k \longrightarrow M \otimes_R N \longrightarrow 0.$$

We have seen in class that when we have a short exact sequence of R-modules, the module in the middle is Noetherian if and only if the other two modules are Noetherian. Since N is Noetherian and a finite sum of Noetherian modules is Noetherian, we obtain that N^k is Noetherian. Applying this to the exact sequence above, we obtain that $\ker(\phi \otimes \mathrm{id}_N)$ and $M \otimes_R N$ are Noetherian, the second being the result we desired.

Exercise 2

We consider V a finite dimensional vector space over a field K, $A_V \longrightarrow V$ a linear transformation and the multiplication:

$$\begin{array}{ccc} K[x] \times V & \longrightarrow & V \\ (f, v) & \longmapsto & f(A)v \end{array}.$$

1. We verify that V is a $K[x]$-module with the multiplication above. First, since V is a finite dimensional K vector space, V has the structure of an abelian group with respect to addition. Hence we only have to verify that given $u, v \in V$ and $f, g \in K[x]$, say $f = \sum_{i=1}^n f_i x^i$, $g = \sum_{i=1}^n g_i x^i$ with $f_i, g_i \in K$ for $i = 1, \ldots, n$:

   (a) We have:

   $$\begin{aligned} f \cdot (u+v) &= f(A)(u+v) = \left(\sum_{i=1}^n f_i A^i\right)(u+v) = \sum_{i=1}^n f_i A^i (u+v) \\ &= \sum_{i=1}^n f_i (A^i u + A^i v) = \sum_{i=1}^n f_i A^i u + \sum_{i=1}^n f_i A^i v = f \cdot u + f \cdot v, \end{aligned}$$

   where we have used that A is K-linear hence $A(u+v) = Au + Av$.

   (b) We have:

   $$\begin{aligned} (f+g) \cdot v &= \left(\sum_{i=1}^n f_i x^i + \sum_{i=1}^n g_i x^i\right) \cdot v = \left(\sum_{i=1}^n (f_i + g_i) x^i\right) \cdot v \\ &= \sum_{i=1}^n (f_i + g_i) A^i v = \sum_{i=1}^n f_i A^i v + \sum_{i=1}^n g_i A^i v = f \cdot v + g \cdot v. \end{aligned}$$

   (c) We have:

   $$\begin{aligned} (fg) \cdot v &= \left(\sum_{i,j} f_i g_j x^{i+j}\right) \cdot v = \sum_{i,j} f_i g_j A^{i+j} v = \sum_{i=1}^n f_i A^i \left(\sum_{j=1}^n g_j A^j v\right) \\ &= \sum_{i=1}^n f_i A^i (g \cdot v) = f \cdot (g \cdot v), \end{aligned}$$

   where we have used that A is K-linear hence its action commutes with $f_i, g_i \in K$ for $i = 1, \ldots, n$.

   (d) We have:

   $$1 \cdot v = \mathrm{id}_V v = v.$$

   This means that indeed V is a $K[x]$-module with the above multiplication.

2. Show that V is a finitely generated torsion $K[x]$-module.

   First, we note that V is a finite dimensional vector space, say $\dim(V) = n$, this means that $V = \langle v_1, \dots, v_n \rangle_K$ where $v_1, \dots, v_n$ generate V and are linearly independent, both over K. This clearly means that $V = \langle v_1, \dots, v_n \rangle_{K[x]}$ since we can generate V with $K \subset K[x]$ (because the multiplication by elements of K as a $K[x]$-module is the same as the usual multiplication by scalars as a K-module) and adding possible coefficients to the linear combinations doesn't change that (however, now we cannot say that $v_1, \dots, v_n$ are linearly independent since there may be a linear combination of the generators with non-zero coefficients in $K[x]$ that is zero). This proves that V is finitely generated as $K[x]$-module.

   To prove that it is a torsion module, let $v \in V$ and define the map:

$$\begin{array}{cccc} \varphi_v & : & K[x] & \longrightarrow & V \\ & & f & \longrightarrow & f \cdot v\text{'} \end{array}$$

   notice that φ_v is a homomorphism of $K[x]$-modules, since given $f, g \in K[x]$, say $f = \sum_{i=1}^{n} f_i x^i$, $g = \sum_{i=1}^{n} g_i x^i$, with $f_i, g_i, \in K$ for $i = 1, \dots, n$, we have:

   (a) $\varphi_v(f + g) = (f + g) \cdot v = f \cdot v + g \cdot v = \varphi_v(f) + \varphi_v(g)$,
   (b) $\varphi_v(fg) = (fg) \cdot v = f \cdot (g \cdot v) = f \cdot \varphi_v(g)$,

   where we have used the properties of the multiplication proved in the section above. This yields the short exact sequence of $K[x]$-modules:

$$0 \longrightarrow \ker(\varphi_v) \xrightarrow{\iota} K[x] \xrightarrow{\varphi_v} \operatorname{im}(\varphi_v) \longrightarrow 0,$$

   where ι denotes the natural inclusion. In particular, this is a short exact sequence of K-modules with K a field, and since every vector space has a basis, all the modules in the sequence are free modules, meaning that the sequence splits. Thus as K vector spaces, we have that $K[x] \cong \ker(\varphi_v) \oplus \operatorname{im}(\varphi_v)$. Notice that $\operatorname{im}(\varphi_v) \subset V$ and we know that a sub-vector space of a finite dimensional vector space is also finite dimensional (with the dimension bounded by the dimension of the vector space containing it), thus $\operatorname{im}(\varphi_v)$ is a finite dimensional K vector space. Moreover, $K[x]$ is an infinite dimensional K vector space. Suppose $\ker(\varphi_v) = \{0\}$, this means that an infinite dimensional K vector space is isomorphic to a finite dimensional K vector space, which is absurd, hence we must have $\ker(\varphi_v) \neq \{0\}$. In particular, there is a non-zero element $f_v \in \ker(\varphi_v)$, that is, $f_v \cdot v = \varphi_v(f_v) = 0$. Hence, we found $f_v \in K[x]$ such that $f_v \cdot v = 0$, that is, v is a torsion element. Since this is true for every $v \in V$, we have that V is a torsion $K[x]$-module.

3. Suppose $K = \mathbb{R}$, $V = \mathbb{R}^2$ and:

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

We want to find a polynomial $q \in \mathbb{R}[x]$ so that $V \cong \mathbb{R}[x]/(q)$ as $\mathbb{R}[x]$-modules. Since we have that $A^2 = -4\mathrm{id}_V$, we foresee that $q = x^2 + 4$.

For this, we first consider $\mathbb{R}^2$ with the usual basis $(1,0)$ and $(0,1)$. We now define the map:

$$\begin{aligned} \psi \; : \; \mathbb{R}[x] &\longrightarrow \mathbb{R}^2 \\ f &\longmapsto f \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{recall} \quad f \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = f(A) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

Notice that $\psi = \varphi_{(1,0)}$ in the notation of the section above, thus in particular ψ is a $\mathbb{R}[x]$-module homomorphism.

Moreover, it is surjective, since $\psi(1) = 1 \cdot (1,0) = (1,0)$ and $\psi(\frac{x}{-2}) = \frac{1}{-2}(0,-2) = (0,1)$, so both elements of the canonical basis of $\mathbb{R}^2$ belong to the image of ψ. The fact that ψ is a $\mathbb{R}[x]$-module homomorphism ensures that $\mathrm{im}(\psi)$ is a $\mathbb{R}[x]$-module (hence linear combinations with coefficients in $\mathbb{R}[x]$ of elements in the image remains in the image), so $\mathbb{R}^2 \subset \mathrm{im}(\psi)$, meaning that $\mathrm{im}(\psi) = \mathbb{R}^2$.

We prove that $\ker(\psi) = (x^2+4)$. Clearly $\psi(x^2+4) = A^2+4\mathrm{id}_V = -4\mathrm{id}_V+4\mathrm{id}_V = 0$, so $(x^2+4) \subset \ker(\psi)$. Suppose $f \in \ker(\psi)$, since we are in $\mathbb{R}[x]$, we use the division algorithm to obtain $f = (x^2+4)h+r$ with $\deg(r) < 2$. Applying ψ to this equality and using the above, we obtain that:

$$0 = \psi(f) = \psi(x^2+4)\psi(h) + \psi(r) = \psi(r),$$

and if we write $r = r_0 + r_1 x$ we obtain that:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = r_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2r_2 \end{bmatrix} \Longrightarrow \begin{cases} r_0 = 0 \\ -2r_1 = 0 \end{cases} \Longrightarrow \begin{cases} r_0 = 0 \\ r_1 = 0 \end{cases},$$

meaning that $r = 0$ and thus $f = (x^2+4)h$, that is $f \in (x^2+4)$ hence $\ker(\psi) \subset (x^2+4)$. This means that indeed $\ker(\psi) = (x^2+4)$.

Finally, we have $V = \mathrm{im}(\psi) \cong \mathbb{R}[x]/\ker(\psi) = \mathbb{R}[x]/(x^2+4)$ by the First Isomorphism Theorem. Setting $q = x^2+4 \in \mathbb{R}[x]$, this is the desired isomorphism of $\mathbb{R}[x]$-modules.

Exercise 3

We use the notation in the sections above, with $A : V \longrightarrow V$ a linear transformation of V a finite dimensional K vector space, say $\dim(V) = n$.

1. We show that there is a unique polynomial $q_A \in K[x]$ of least degree for which $q_A(A) = 0$, and we determine its expression in terms of the elementary divisors of V as $K[x]$-module.

   For this, consider the set $\{\deg(f) : 0 \neq f \in K[x],\, f \cdot v = 0\,\forall v \in V\} \subset \mathbb{N}$. We note that this set is non empty since given a K basis $v_1, \ldots, v_n$ of V, this being a torsion $K[x]$-module assures us that there are polynomials $f_{v_1}, \ldots, f_{v_n} \in K[x]$ such that $f_{v_i} \cdot v_i = 0$ for $i = 1, \ldots, n$. In particular for a general $v \in V$, say $v = \sum_{i=1}^n k_i v_i$ for $k_1, \ldots, k_n \in K$ we have:

   $$f_{v_1} \cdots f_{v_n} \cdot v = \sum_{i=1}^n k_i (f_{v_1} \cdots f_{v_n}) \cdot v_i = \sum_{i=1}^n k_i (f_{v_1} \cdots \hat{f_{v_i}} \cdots f_{v_n} f_{v_i}) \cdot v_i = 0$$

   because V is a $K[x]$-module. Hence the considered set contains $\deg(f_{v_1} \cdots f_{v_n})$. Now, since the minimum degree attainable for a polynomial annihilating every vector in V is 1, because the identity doesn't annihilate any non-zero vector, and the degrees have discrete values because they belong in $\mathbb{N}$, we have that the infimum of the considered set is attained, that is, $\inf\{\deg(f) : 0 \neq f \in K[x],\, f \cdot v = 0\,\forall v \in V\}$ is attained by some polynomial, say $q \in K[x]$. This proves existence of q, a polynomial of least degree such that $q \cdot v = 0$ for every $v \in V$. Now, dividing all the coefficients in q by its leading coefficient, we obtain $q_A \in K[x]$ a monic polynomial of least degree, and since q_A differs of q by multiplication of a scalar in K, we still have that that $q_A \cdot v = 0$ for every $v \in V$. Moreover, this last property implies that $q_A(A) = 0$ as a matrix: suppose that $q_A(A) \neq 0$, this means that using the above K-basis $v_1, \ldots, v_n$ for both V in the domain and target, $q_A(A)$ has a non-zero entry, say $a_{ij} \in K$, but this means that $q_A(A)v_j$ has a_{ij} (which is non-zero) as coefficient for v_i, a contradiction with $q_A \cdot v_j = 0$. Note that this is a general proof, we actually proved that $f \in K[x]$ with $f \cdot v = 0$ implies $f(A) = 0$. Finally, suppose there is $q'_A \in K[x]$, $q'_A \neq q_A$, a monic polynomial of least degree for which $q'_A(A) = 0$, then $q_A - q'_A$ is a non-zero polynomial of degree strictly less than both q_A and q'_A for which $(q_A - q'_A) \cdot v = (q_A - q'_A)(A)v = q_A(A)v - q'_A(A)v = 0$, which is a contradiction with the minimality of the degree. This means that q_A is unique. Thus $q_A \in K[x]$ is the unique monic polynomial of least degree for which $q_A(A) = 0$, as desired.

   To describe q_A in terms of the elementary divisors, we use that V is finitely generated as $K[x]$-module, say with generators $e_1, \ldots, e_m$. Let F be the free $K[x]$-module with basis $e_1, \ldots, e_m$, as noted multiple times this means that the map $\phi : F \longrightarrow M$ given by $\phi(e_i) = e_i$ for $i = 1, \ldots, m$ is a surjective $K[x]$-module homomorphism. Thus we have the exact sequence of $K[x]$-modules:

   $$0 \longrightarrow \ker(\phi) \xrightarrow{\iota} F \xrightarrow{\phi} V \longrightarrow 0,$$

where ι denotes the natural inclusion. Applying the First Isomorphism Theorem we have that $V \cong F/\ker(\phi)$, and by the Elementary Divisors Theorem, using that V is a torsion $K[x]$-module, we obtain that $V \cong K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r)$ with $q_1, \ldots, q_r \in K[x]$ and $q_1 | \cdots | q_r$ (notice that the degree of all the elementary divisors is greater than 1 because K is a field, in particular it has no zero divisors. This will be useful later). Now, let $v \in V$ considered as an element of the direct sum, we have that $q_r \cdot v = 0$ since q_r annihilates every component of the direct sum in virtue of $q_1 | \cdots | q_r$ (in particular by the above $q_r(A) = 0$). The division algorithm gives us that $q_A = q_r h + r$ with $\deg(q_r) < \deg(r)$, and since $0 = q_A(A) = q_r(A)h(A) + r(A) = r(A)$, and $\deg(q_A) = \deg(q_r h) = \deg(q_r) + \deg(h) < \deg(r)$ the minimality of the degree of q_A guarantees that $r = 0$. Moreover, if $\deg(h) > 0$ we have that $\deg(q_A) < \deg(q_r)$, thus again by the minimality of the degree of q_A we have $h \in K$. Since q_A is monic, the only possible value for h is the inverse of the leading coefficient of q_r, that is, q_A is the monic polynomial that arises from q_r, this is the relation that we desired.

2. Let $p_A(x) = \det(x \cdot \mathrm{id}_V - A) \in K[x]$ the characteristic polynomial of A. We want to relate p_A to the elementary divisors of A. As we have seen above, we can write $V \cong K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r)$ with $q_1, \ldots, q_r \in K[x]$ and $q_1 | \cdots | q_r$ (we can assume that they are monic), say this isomorphism is given by ψ. Since we set $\dim_K(V) = n$ and the dimension over K of each $K[x]/(q_i)$ is the degree of q_i, we have that $n = \dim_K(V) = \deg(q_1) + \cdots + \deg(q_r)$. Moreover, since $\deg(p_A) = n$, we foresee that we will have that $p_A = q_1 \cdots q_r$. To prove this, we will have to find an explicit expression of the action by A on $K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r)$. Notice:

$$\begin{array}{ccc} V & \xrightarrow{\quad\quad A \quad\quad} & V \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi} \\ K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r) & \overset{B}{\dashrightarrow} & K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r) \end{array}$$

where we have defined $B = \psi \circ A \circ \psi^{-1}$ (where the compositions are in fact multiplication of matrices), which translates the action on A as desired. We are interested in computing p_A, but notice that:

$$\begin{aligned} p_B &= \det(x \cdot \mathrm{id}_V - B) = \det(x \cdot \mathrm{id}_V - \psi \cdot A \cdot \psi^{-1}) \\ &= \det(\psi \cdot (x \cdot \mathrm{id}_V - A) \cdot \psi^{-1}) = \det(\psi) \det(x \cdot \mathrm{id}_V - A) \det(\psi^{-1}) \\ &= \det(\psi) \det(\psi^{-1}) \det(x \cdot \mathrm{id}_V - A) = \det(\psi \cdot \psi^{-1}) p_A = p_A, \end{aligned}$$

thus we will compute p_B.

Since the composition of $K[x]$-module homomorphisms is a $K[x]$-module homomorphism, in particular B is linear and we have that restricting the domain to one summand of the direct sums, the target is the same summand: $B_i = B|_{K[x]/(q_i)} : K[x]/(q_i) \longrightarrow K[x]/(q_i)$ for every $i = 1, \ldots, r$. Hence $B = B_1 \oplus \cdots \oplus B_r : K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r) \longrightarrow K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r)$. This means that considering the canonical K basis for each summand in $K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r)$ and writing B as a matrix from this space to itself, it has the form of r square matrices on the diagonal:

$$B = \begin{bmatrix} \boxed{B_1} & & \\ & \ddots & \\ & & \boxed{B_r} \end{bmatrix}$$

where B_i for $i = 1, \ldots, r$ represents the matrix of the i-th summand. In particular:

$$\begin{aligned} p_B &= \det(x \cdot \mathrm{id}_V - A) = \begin{vmatrix} \boxed{x \cdot \mathrm{id}_V - B_1} & & \\ & \ddots & \\ & & \boxed{x \cdot \mathrm{id}_V - B_r} \end{vmatrix} \\ &= \det(x \cdot \mathrm{id}_V - B_1) \cdots \det(x \cdot \mathrm{id}_V - B_r) = p_{B_1} \cdots p_{B_r}, \end{aligned}$$

where we have used the Linear Algebra result that the determinant can be computed by expanding along the blocs. Hence it is enough to compute the p_{B_i} for $i = 1, \ldots, n$.

For this, we need to know what the action of B is explicitly. Given an element in $K[x]/(q_1) \oplus \cdots \oplus K[x]/(q_r)$, we can write it uniquely as $\psi(v)$ for a certain $v \in V$, and now $B(\psi(v)) = \psi \circ A \circ \psi^{-1}(\psi(v)) = \psi(Av) = \psi(x \cdot v) = x \cdot \psi(v)$ because ψ is a $K[x]$-module isomorphism. Hence the action of B is simply multiplying by $x \in K[x]$. This translates to every direct summand, hence for every $i = 1, \ldots, r$ letting $q_i = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ (as noted in the section above, here we must have $d > 1$), we have that $B_i : K[x]/(q_i) \longrightarrow K[x]/(q_i)$ acts on the canonical K basis of $K[x]/(q_i)$ as $B(1) = x, \ldots, B(x^{d-2}) = x^{d-1}, B(x^{d-1}) = x^d = -a_{d-1}x^{d-1} - \cdots - a_0$, that is:

$$B_i = \begin{bmatrix} 0 & & \cdots & & -a_0 \\ 1 & 0 & \cdots & & -a_1 \\ & & \ddots & & \vdots \\ & & \ddots & 0 & -a_{d-2} \\ & & & 1 & -a_{d-1} \end{bmatrix},$$

that is, the matrix with the coefficients of q_i with changed sign in the last column, zeroes on the rest of the diagonal and ones under the diagonal. As a small prelude, we compute the following determinant by induction: we prove that, in general:

$$\begin{vmatrix} x & & \cdots & & a_0 \\ -1 & x & \cdots & & a_1 \\ & & \ddots & & \vdots \\ & & \ddots & x & a_{d-2} \\ & & & -1 & x + a_{d-1} \end{vmatrix} = a_0 + \cdots + a_{d-1}x^{d-1} + x^d.$$

For this, we proceed by induction on the dimension of the matrix. When it is a 1×1 matrix, we clearly have $\det(x) = x$. Now suppose this is true for $(d-1)\times(d-1)$ matrices and $d > 1$, for $d \times d$ matrices we have:

$$\begin{aligned}\begin{vmatrix} x & & \cdots & & a_0 \\ -1 & x & \cdots & & a_1 \\ & & \ddots & & \vdots \\ & & \ddots & x & a_{d-2} \\ & & & -1 & x+a_{d-1} \end{vmatrix} &= x\cdot \begin{vmatrix} x & & \cdots & & a_1 \\ -1 & x & \cdots & & a_2 \\ & & \ddots & & \vdots \\ & & \ddots & x & a_{d-2} \\ & & & -1 & x+a_{d-1} \end{vmatrix} \\ &+ (-1)^{d-1}a_0 \begin{vmatrix} -1 & x & \cdots & & 0 \\ 0 & -1 & \cdots & & 0 \\ & & \ddots & & \vdots \\ & & \ddots & -1 & x \\ & & & 0 & -1 \end{vmatrix} \\ &= x(a_1 + \cdots + a_{d-1}x^{d-2} + x^{d-1}) \\ &+ (-1)^{d-1}(-1)^{d-1}a_0 = a_0 + \cdots + a_{d-1}x^{d-1} + x^d \end{aligned}$$

where we have first expanded along the first row and then used the induction hypothesis. This is clearly what we claimed. We now have that:

$$p_{B_i} = \begin{vmatrix} x & & \cdots & & a_0 \\ -1 & x & \cdots & & a_1 \\ & & \ddots & & \vdots \\ & & \ddots & x & a_{d-2} \\ & & & -1 & x+a_{d-1} \end{vmatrix} = a_0 + \cdots + a_{d-1}x^{d-1} + x^d = q_i$$

meaning that $p_B = p_{B_1} \cdots p_{B_r} = q_1 \cdots q_r$, the relation with the elementary divisors of V that we desired.

3. We want to see that $p_A(A) = 0$. By the section above, we know that q_A divides p_A, that is, there is $h \in K[x]$ such that $p_A = q_A h$. This means that when we apply this to A we obtain $p_A(A) = q_A(A)h(A) = 0$ since $q_A(A) = 0$, the desired result.