Algebra II - Homework 7

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Exercise 1

Let K be a field, E/K and algebraic extension, R an integral domain with $K \subset R \subset E$. We want to see that R is a field. Since it is already an integral domain, all the algebraic axioms from the definition of a field via the relation between their elements follow, except the one that every non-zero $r \in R$ has an inverse. Given r, we will find $s \in R$ with $rs = 1$.

Since $r \in R \subset E$, there is an element $s \in E$ with $rs = 1$. Moreover, since E/K is algebraic, there is a polynomial $f \in K[x]$ that annihilates s, say $f = a_n x^n + \cdots + a_0$ with $a_i \in K$ for $i = 1, \ldots, n$. Hence:

$$\begin{aligned}
0 &= a_n s^n + \cdots + a_1 s + a_0 \\
0 &= r^{n-1}(a_n s^n + \cdots + a_1 s + a_0) \\
0 &= a_n (rs)^{n-1} s + \cdots + a_1 rsr^{n-2} + a_0 r^{n-1} \\
0 &= a_n s + \cdots + a_1 r^{n-2} + a_0 r^{n-1} \\
a_n s &= -a_{n-1} - \cdots - a_0 r^{n-1} \\
s &= -(a_{n-1} + \cdots + a_0 r^{n-1})/a_n
\end{aligned}$$

and since K is a field, we have that $-1/a_n \in K \subset R$, and the right hand side is sum and multiplication of elements in the ring R, so in fact $s \in R$ with $rs = 1$, as desired.

Exercise 2

Let L/K be an extension of fields, let $\alpha \in L$ algebraic over K with $[K(\alpha) : K]$ odd. We prove that $[K(\alpha^2) : K]$ is also odd and $K(\alpha) = K(\alpha^2)$.

Notice that by the field tower $K \subset K(\alpha^2) \subset K(\alpha)$ we have that $[K(\alpha) : K] = [K(\alpha) : K(\alpha^2)][K(\alpha^2) : K]$. Since $x^2 - \alpha^2 \in K(\alpha^2)[x]$ annihilates α, we have that $[K(\alpha) : K(\alpha^2)] \leq 2$. Since multiplying by 2 would result in $[K(\alpha) : K]$ even, we need to have $[K(\alpha) : K(\alpha^2)] = 1$ and $[K(\alpha^2) : K] = [K(\alpha) : K]$ is odd.

Notice that for any field extension F/E, we have $[E : F] = 1$ if and only if $E = F$. This follows because $\{1\}$ is linearly independent in E over F (because E is a field), so it is a basis, hence $F = E \cdot 1 = E$. Notice how this proves both directions. Since we have that $[K(\alpha) : K(\alpha^2)] = 1$, we must have $K(\alpha) = K(\alpha^2)$, as desired.

Exercise 3

Let $\alpha = \sqrt{3} + \sqrt{5} \in \mathbb{R}$. We determine $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and $\mathrm{Irr}(\alpha, \mathbb{Q}, x) \in \mathbb{Q}[x]$.

Notice how:

$$\alpha^2 = 8 + 2\sqrt{15}, \quad \alpha^3 = 18\sqrt{3} + 14\sqrt{5}, \quad \alpha^4 = 124 + 32\sqrt{15}$$

meaning that $f = x^4 - 16x^2 + 4 \in \mathbb{Q}[x]$ annihilates α. Since the solutions of f are $s^2 = 8 \pm \sqrt{5}$, that is, $s = \pm\sqrt{8 \pm \sqrt{5}}$, we have that $s, s^2 \notin \mathbb{Q}$. Hence f cannot decompose in polynomials of degree 1 or 2, meaning that in fact it cannot decompose in polynomials of degree 3 (because the other factor would have degree 1), which implies that f is irreducible. Since f is already monic, we have that $\mathrm{Irr}(\alpha, \mathbb{Q}, x) = x^4 - 16x^2 + 4$ and that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg(\mathrm{Irr}(\alpha, \mathbb{Q}, x)) = 4$.

Exercise 4

Let K be a field of characteristic different from 2, L algebraically closed field containing K. During this exercise we will use that for every $u, v \in L$ we have $\sqrt{u}\sqrt{v} = \sqrt{uv}$, which is due to both being choices of square roots of the element $uv \in K$.

1. Let $a, b \in K$ with $x^2 - a$ and $x^2 - b$ both irreducible in $K[x]$. Denote $\sqrt{a}$, $\sqrt{b}$ choices of square roots of a, b respectively. We will abuse that $[K(\sqrt{a}, \sqrt{b}) : K] = [K(\sqrt{a}, \sqrt{b}) : K(\sqrt{b})][K(\sqrt{b}) : K]$.

   If ab is a square in K, that is, there is $t \in K$ with $t^2 = ab$, otherwise said, $t = \sqrt{ab}$ is a choice of a square root in K, we have that in $K(\sqrt{b})$:

   $$\left(\frac{t}{\sqrt{b}}\right)^2 = \frac{t^2}{b} = \frac{ab}{b} = a,$$

   and thus $t/\sqrt{b} \in K(\sqrt{b})$ is a choice of square root of a, that is $\sqrt{a} = \pm t/\sqrt{b} \in K(\sqrt{b})$, and thus $K(\sqrt{a}, \sqrt{b}) = K(\sqrt{b})$. As noted in a section above, this means $[K(\sqrt{a}, \sqrt{b}) : K(\sqrt{b})] = 1$ and $[K(\sqrt{a}, \sqrt{b}) : K] = [K(\sqrt{b}) : K] = 2$ because $x^2 - b$ is monic and irreducible in $K[x]$.

   If ab is not a square in K, we now prove that $x^2 - a$ is irreducible in $K(\sqrt{b})$. Suppose not, that is, $x^2 - a$ has a solution in $K(\sqrt{b})$, otherwise said, $\sqrt{a} \in K(\sqrt{b})$. Since we have $K(\sqrt{b}) \cong \{r + s\sqrt{b} : r, s \in K\}$ in virtue of $K(\sqrt{b}) \cong K[x]/(x^2 - b)$, this means that we can write $\sqrt{a} = r + s\sqrt{b}$ for certain $r, s \in K$. Taking the square on both sides we obtain $a = r^2 + s^2 b + 2rs\sqrt{b}$, meaning that we must have $rs = 0$. Since K is a field, this means that either $r = 0$ or $s = 0$. Now $s = 0$ implies $\sqrt{a} = r \in K$, contradicting that $x^2 - a$ is irreducible in $K[x]$. Now $r = 0$ implies that $\sqrt{a} = s\sqrt{b}$ and thus $(sb)^2 = s^2 bb = ab$, meaning that ab is a square in K, a contradiction. Hence $x^2 - a$ is indeed irreducible and monic, so $[K(\sqrt{a}, \sqrt{b}) : K(\sqrt{b})] = 2$ and $[K(\sqrt{a}, \sqrt{b}) : K] = 2 \cdot 2 = 4$.

2. Let $c, d \in K$ with d not a square in K. Fix $\sqrt{b} \in L$ a square root of d in L. Let $\alpha = \sqrt{c + \sqrt{d}}$ be a choice of square root of $c + \sqrt{d}$ in L. We prove that there are $a, b \in K$ with $\alpha = \sqrt{a} + \sqrt{b}$ if and only if $c^2 - d$ is a square in K.

   $\Rightarrow$) Using $\alpha = \sqrt{c + \sqrt{d}}$ we have that $\alpha^2 = c + \sqrt{d}$ and $\alpha^4 = c^2 + d + 2c\sqrt{d}$, so $f = x^4 - 2cx^2 + (c^2 - d) \in K[x]$ annihilates α. Using $\alpha = \sqrt{a} + \sqrt{b}$ we have that $\alpha^2 = a + b + 2\sqrt{ab}$ and $\alpha^4 = 6ab + a^2 + b^2 + 4a\sqrt{ab} + 4b\sqrt{ab}$, so $f = x^4 - 2(a + b)x^2 + (a - b)^2 \in K[x]$ annihilates α. Now, we have that $[K(\alpha) : K] = [K(\alpha) : K(\sqrt{d})][K(\sqrt{d}) : K]$ and we know that $[K(\sqrt{d}) : K] = 2$ because d is not a square in K. Moreover, $h = x^2 - c - \sqrt{d} \in K(\sqrt{d})$ annihilates α, so $[K(\alpha) : K(\sqrt{d})]$ must be 1 or 2.

   If $[K(\alpha) : K(\sqrt{d})] = 2$ we have that $[K(\alpha) : K] = 2 \cdot 2 = 4$, meaning that $f = g$ by uniqueness of the monic polynomial of minimal degree annihilating α. Hence comparing coefficients we find that $c^2 - d = (a - b)^2$ is a square in K.

If $[K(\alpha) : K(\sqrt{d})] = 1$ we have that $K(\alpha) = K(\sqrt{d})$ and thus $\alpha \in K(\sqrt{d})$. By the noted above, this means that we can find $r, s \in K$ with $\alpha = r + s\sqrt{d}$. Squaring both sides we obtain that $c + \sqrt{d} = r^2 + s^2 d + 2rs\sqrt{d}$, so comparing coefficients we must have that $r^2 + s^2 d = c$ and $2rs = 1$. Hence $c^2 - d = s^4 d^2 + (rs - 1)d + r^4$, otherwise said the polynomial $h = s^4 x^2 + (rs - 1)x + r^4 \in K[x]$ satisfies $h(d) = c^2 - d$. The discriminant of h is:

$$(rs-1)^2 - 4r^4s^4 = (2r^2s^2 - 1)^2 - 4r^4s^4 = 4r^4s^4 + 1 - 4r^2s^2 - 4r^4s^4 = 0,$$

where we used multiple times that $2rs = 1$. Thus the root of h is $(1 - rs)/2s^4$ and we can write $h = s^4(x - (1 - rs)/2s^4)^2 = (s^2(x - (1 - rs)/2s^4))^2$, meaning that $c^2 - d = h(d) = (s^2(d - (1 - rs)/2s^4))^2$ with $s^2(d - (1 - rs)/2s^4) \in K$, thus $c^2 - d$ is again a square in K.

$\Leftarrow$) Suppose we can write $c^2 - d = t^2$ for certain $t \in K$, in particular $d = c^2 - t^2 = (c+t)(c-t)$. We want $a, b \in K$ with $\sqrt{a} + \sqrt{b} = \sqrt{c + \sqrt{d}} = \sqrt{c + \sqrt{(c+t)(c-t)}}$, so squaring we must have $a + b + 2\sqrt{ab} = c + \sqrt{(c+t)(c-t)}$. For this, one would expect to have $a + b = c$ and $2\sqrt{ab} = \sqrt{(c+t)(c-t)}$, so it looks natural to propose $a = (c+t)/2$ and $b = (c-t)/2$. They indeed satisfy:

$$(\sqrt{a} + \sqrt{b})^2 = \frac{c+t}{2} + \frac{c-t}{2} + 2\sqrt{\frac{c+t}{2}}\sqrt{\frac{c-t}{2}} = c + \sqrt{d}.$$

Since we have $a, b \in K$, we obtained the desired elements.

3. Show that $\mathbb{Q}(\sqrt{3+\sqrt{5}})/\mathbb{Q}$ is a biquadratic extension and find $a, b \in \mathbb{Q}$ with $\sqrt{3+\sqrt{5}} = \sqrt{a} + \sqrt{b}$.

   We rename $\alpha = \sqrt{3+\sqrt{5}}$. In this particular case we have $c = 3$ and $d = 5$, and since $c^2 - d = 9 - 5 = 4 = 2^2$ is a square in K, by the above we are guaranteed that $a = 5/2, b = 1/2 \in \mathbb{Q}$ satisfy $\sqrt{3+\sqrt{5}} = \sqrt{5/2} + \sqrt{1/2} = \sqrt{a} + \sqrt{b}$.

   Moreover $f = x^4 - 2cx^2 + (c^2 - d) = x^4 - 6x^2 + 4 \in \mathbb{Q}[x]$ annihilates α. Since the solutions of f are $s^2 = 3 \pm \sqrt{5}$, that is, $s = \pm\sqrt{3 \pm \sqrt{5}}$, we have that $s, s^2 \notin \mathbb{Q}$. Hence as reasoned above, f cannot decompose in polynomials of degree 1 or 2, meaning that in fact it cannot decompose in polynomials of degree 3 (because the other factor would have degree 1), which implies that f is irreducible in $\mathbb{Q}$. Since f is already monic, we have that $f = \mathrm{Irr}(\alpha, \mathbb{Q}, x)$ and that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg(\mathrm{Irr}(\alpha, \mathbb{Q}, x)) = 4$. Moreover, since $(x^2 - 5/2)(x^2 - 1/2) \in \mathbb{Q}[x]$ is monic of degree 4 and annihilates both $\sqrt{5/2}, \sqrt{1/2}$, we have that $[\mathbb{Q}(\sqrt{5/2}, \sqrt{1/2}) : \mathbb{Q}] \leq 4$. Considering $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\sqrt{a} + \sqrt{b}) \subset \mathbb{Q}(\sqrt{a}, \sqrt{b})$, the first element having dimension 4 over $\mathbb{Q}$ and the last having at most dimension 4 over $\mathbb{Q}$ means that in fact $[\mathbb{Q}(\sqrt{5/2}, \sqrt{1/2}) : \mathbb{Q}] = 4$, all the above inclusions are equalities and $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{a}, \sqrt{b})$. Hence the extension $\mathbb{Q}(\sqrt{5/2}), \sqrt{1/2})/\mathbb{Q}$ is biquadratic, as desired.