Introduction to Commutative and Homological Algebra - Homework 1

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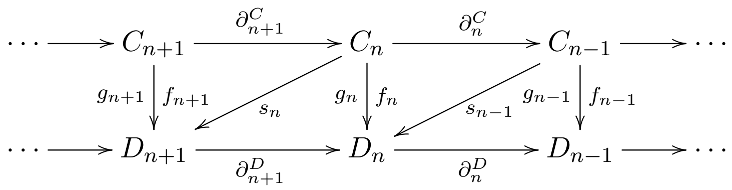
Exercise 1

We prove that chain homotopy is an equivalence relation. Recall that given two chain maps $f., g. : C. \longrightarrow D.$, we say that they are chain homotopic if there are R module homomorphisms $s_n : C_n \longrightarrow D_{n+1}$ for each $n \in \mathbb{Z}$ such that $f_n - g_n = s_{n-1}\partial_n^C + \partial_{n+1}^D s_n$:



We now have:

1. Reflexive: cleary $f. \sim f.$ by using $s_n \equiv 0$ since $f_n - f_n = 0$ for all $n \in \mathbb{Z}$.

2. Symmetric: if $f. \sim g.$ by s_n, then $g. \sim f.$ by $-s_n$ since $g_n - f_n = -(f_n - g_n) = (-s_{n-1})\partial_n^C + \partial_{n+1}^D(-s_n)$ for all $n \in \mathbb{Z}$.

3. Transitive: if $f. \sim g.$ by s_n and $g. \sim h.$ by t_n, then $f. \sim h.$ by $s_n + t_n$ since $f_n - h_n = (f_n - g_n) + (g_n - h_n) = s_{n-1}\partial_n^C + \partial_{n+1}^D s_n + t_{n-1}\partial_n^C + \partial_{n+1}^D t_n = (s_{n-1} + t_{n-1})\partial_n^C + \partial_{n+1}^D(s_n + t_n)$ for all $n \in \mathbb{Z}$.

and thus this is an equivalence relation.

Exercise 2

Let B be a left R-module and x is not a zero divisor of R. We have a projective resolution of R/xR as:

$$0 \longrightarrow R \xrightarrow{x\cdot} R \xrightarrow{\pi} R/xR \longrightarrow 0$$

where we usa that x is not a zero divisor to obtain injectivity of the map $x\cdot$. Now applying $- \otimes_R B$ and truncating the last term we obtain:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R \otimes_R B & \xrightarrow{(x\cdot)\otimes 1_B} & R \otimes_R B & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & B & \xrightarrow{x\cdot} & B & \longrightarrow & 0 \end{array}$$

where we used the isomorphism given by:

$$\begin{array}{ccc} R \otimes_R B & \longrightarrow & B \\ r \otimes b & \longmapsto & rb \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \longrightarrow & R \otimes_R B \\ b & \longmapsto & 1 \otimes b \end{array}$$

so that clearly:

$$b \overset{\cong}{\longmapsto} 1 \otimes b \overset{(x\cdot)\otimes 1_B}{\longmapsto} x \otimes b \overset{\cong}{\longmapsto} xb$$

is the resulting map in the second row. Thus:

$$\begin{aligned} \mathrm{Tor}_0^R(R/xR, B) &= B/xB, \\ \mathrm{Tor}_1^R(R/xR, B) &= \mathrm{Ker}(x\cdot)/\mathrm{Im}(0) \cong \mathrm{Ker}(x\cdot) = \{b \in B : xb = 0\}, \\ \mathrm{Tor}_n^R(R/xR, B) &= 0 \text{ for } n \geq 2. \end{aligned}$$

the desired result.

Exercise 3

Let A, A', B, B' be left R-modules. For the first relation, we choose projective resolutions for A and A':

$$\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

$$\cdots \longrightarrow P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{\epsilon'} A' \longrightarrow 0$$

which means that adding them up via a direct sum and using the natural induced maps component-wise, we obtain the projective resolution of $A \oplus A'$:

$$\cdots \longrightarrow P_1 \oplus P'_1 \xrightarrow{d_1 \oplus d'_1} P_0 \oplus P'_0 \xrightarrow{\epsilon \oplus \epsilon'} A \oplus A' \longrightarrow 0$$

and the last element that we will use is that in any category $\mathcal{C}$ we have $\mathrm{Hom}_{\mathcal{C}}(\coprod_{i \in I} A_i, B) \cong \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(A_i, B)$ for $A_i, B \in \mathcal{C}$ for all $i \in I$. In the particular case of the category of R-modules, and in our particular case where everything is finite, this translates to $\mathrm{Hom}_R(A \oplus A', B) \cong \mathrm{Hom}_R(A, B) \oplus \mathrm{Hom}_R(A', B)$. This means that applying $\mathrm{Hom}_R(-, B)$ to the last projective resolution we obtain (in a general element of the complex):

$$\begin{array}{ccc}
\mathrm{Hom}_R(P_n, B) \oplus \mathrm{Hom}_R(P'_n, B) & \xrightarrow{d^*_{n+1} \oplus d'^*_{n+1}} & \mathrm{Hom}_R(P_{n+1}, B) \oplus \mathrm{Hom}_R(P'_{n+1}, B) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_R(P_n \oplus P'_n, B) & \xrightarrow{(d_{n+1} \oplus d'_{n+1})^*} & \mathrm{Hom}_R(P_{n+1} \oplus P'_{n+1}, B)
\end{array}$$

where the first row is the sum of the individual complexes obtained from A and A' independently. This means that:

$$\begin{aligned}
Ext^n_R(A \oplus A', B) &= H^n(\mathrm{Hom}_R(P. \oplus P'., B)) \cong H^n(\mathrm{Hom}_R(P., B) \oplus \mathrm{Hom}_R(P'., B)) \\
&= \mathrm{Ker}(d^*_{n+1} \oplus d'^*_{n+1}) / \mathrm{Im}(d^*_{n+1} \oplus d'^*_{n+1}) \\
&\cong \mathrm{Ker}(d^*_{n+1}) \oplus \mathrm{Ker}(d'^*_{n+1}) / \mathrm{Im}(d^*_{n+1}) \oplus \mathrm{Im}(d'^*_{n+1}) \\
&\cong \mathrm{Ker}(d^*_{n+1}) / \mathrm{Im}(d^*_{n+1}) \oplus \mathrm{Ker}(d'^*_{n+1}) / \mathrm{Im}(d'^*_{n+1}) \\
&\cong \mathrm{Ext}^n_R(A, B) \oplus \mathrm{Ext}^n_R(A', B)
\end{aligned}$$

where we have only used basic relations between the direct sum of modules and the quotient of modules and the fact that since $d^*_{n+1} \oplus d'^*_{n+1}$ is a map with two components that do not interact among them, its kernel is just the direct sum of each kernel.

Doing basically the same thing, we will obtain the second relation. We choose a projective resolution for A:

$$\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

which means that applying $\mathrm{Hom}_R(-,B)$, $\mathrm{Hom}_R(-,B')$ and $\mathrm{Hom}_R(-,B\oplus B')$ we obtain the three truncated complexes:

$$\cdots \longrightarrow \mathrm{Hom}_R(P_n,B) \xrightarrow{d^*_{n+1}} \mathrm{Hom}_R(P_{n+1},B) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Hom}_R(P_n,B') \xrightarrow{d^*_{n+1}} \mathrm{Hom}_R(P_{n+1},B') \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Hom}_R(P_n,B\oplus B') \xrightarrow{d^*_{n+1}} \mathrm{Hom}_R(P_{n+1},B\oplus B) \longrightarrow \cdots$$

adding the first two up via a direct sum, as before we can using the natural induced maps component-wise to build a complex, and the last element that we will use is that in any category $\mathcal{C}$ we have $\mathrm{Hom}_{\mathcal{C}}(A,\prod_{i\in I} B_i)\cong\prod_{i\in I}\mathrm{Hom}_{\mathcal{C}}(A,B_i)$ for $A_i,B\in\mathcal{C}$ for all $i\in I$. In the particular case of the category of R-modules, and in our particular case where everything is finite, this translates to $\mathrm{Hom}_R(A,B\oplus B')\cong\mathrm{Hom}_R(A,B)\oplus\mathrm{Hom}_R(A,B')$. We obtain:

$$\begin{array}{ccc}
\mathrm{Hom}_R(P_n,B)\oplus\mathrm{Hom}_R(P_n,B') & \xrightarrow{d^*_{n+1}\oplus d^*_{n+1}} & \mathrm{Hom}_R(P_{n+1},B)\oplus\mathrm{Hom}_R(P'_{n+1},B) \\
\downarrow\cong & & \downarrow\cong \\
\mathrm{Hom}_R(P_n,B\oplus B') & \xrightarrow{\quad d^*_{n+1}\quad} & \mathrm{Hom}_R(P_{n+1},B\oplus B')
\end{array}$$

The notation is quite unfortunate since every map is d^*_{n+1} (since every map really does just what d^*_{n+1} should do). To clarify this, in the following any reference to this will be in the context of the row above, and we have to keep track of what are the domain and range depending on the side of the direct sum where this map is. We have:

$$\begin{aligned}
Ext^n_R(A,B\oplus B') &= H^n(\mathrm{Hom}_R(P.,B\oplus B'))\cong H^n(\mathrm{Hom}_R(P.,B)\oplus\mathrm{Hom}_R(P.,B')) \\
&= \mathrm{Ker}(d^*_{n+1}\oplus d^*_{n+1})/\mathrm{Im}(d^*_{n+1}\oplus d^*_{n+1}) \\
&\cong \mathrm{Ker}(d^*_{n+1})\oplus\mathrm{Ker}(d'^*_{n+1})/\mathrm{Im}(d^*_{n+1})\oplus\mathrm{Im}(d'^*_{n+1}) \\
&\cong \mathrm{Ker}(d^*_{n+1})/\mathrm{Im}(d^*_{n+1})\oplus\mathrm{Ker}(d^*_{n+1})/\mathrm{Im}(d^*_{n+1}) \\
&\cong \mathrm{Ext}^n_R(A,B)\oplus\mathrm{Ext}^n_R(A,B')
\end{aligned}$$

where again we have only used basic relations between the direct sum of modules and the quotient of modules and the fact that since $d^*_{n+1}\oplus d^*_{n+1}$ is a map with two components that do not interact among them, its kernel is just the direct sum of each kernel.

Exercise 4

We prove that an R-module Q is injective if and only if $\mathrm{Ext}^1_R(R/I, Q) = 0$ for all left ideals I.

$\Rightarrow$) Suppose Q is injective, we saw in class that this is equivalent to $\mathrm{Ext}^1_R(A, B) = 0$ for every left R-module A. Since R/I is a left R-module under the natural multiplication, we obtain $\mathrm{Ext}^1_R(R/I, B) = 0$.

$\Leftarrow$) Suppose $\mathrm{Ext}^1_R(R/I, B) = 0$ for all left ideals I. Fix one such I, we have the short exact sequence:

$$0 \longrightarrow I \xrightarrow{\iota} R \xrightarrow{\pi} R/I \longrightarrow 0$$

where ι is the natural inclusion and π is the natural projection. Applying the Second Long Exact Sequence for Ext to this, we obtain:

$$0 \longrightarrow \mathrm{Hom}_R(R/I, Q) \longrightarrow \mathrm{Hom}_R(R, Q) \xrightarrow{\iota^*} \mathrm{Hom}_R(I, Q) \longrightarrow \mathrm{Ext}^1_R(R/I, Q) \cong 0$$

meaning that ι^* is surjective. This means that given $f : I \longrightarrow Q$ an R-module homomorphism, there is a $g : R \longrightarrow Q$ an R-module homomorphism with $\iota^*(g) = f$. This means that for every $x \in I$ we have $f(x) = \iota^*(g)(x) = g(\iota(x)) = g(x)$ and then g indeed extends I to R. Since I was ideal of R, by Baer's Criterion we obtain that Q is injective.

Exercise 5

Let k be a field, $R = k[x]$ and I a non zero proper ideal of R. We first note that R is principal, meaning that there is an element $a \in R$ such that $I = (a)$. Hence R/I has a projective resolution:

$$0 \longrightarrow R \xrightarrow{a\cdot} R \xrightarrow{\pi} R/I \longrightarrow 0$$

because $\mathrm{Ker}(a\cdot) = \{r \in R : ar = 0\} = \{0\}$ since R has no zero divisors.

To obtain Tor, we apply $- \otimes_R R/I$ and truncate the complex, obtaining:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R \otimes_R R/I & \xrightarrow{(a\cdot)\otimes 1_{R/I}} & R \otimes_R R/I & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & R/I & \xrightarrow[0]{a\cdot} & R/I & \longrightarrow & 0 \end{array}$$

where we use the aforementioned isomorphism and hence:

$$\overline{r} \overset{\cong}{\longmapsto} 1 \otimes \overline{r} \overset{(a\cdot)\otimes 1_{R/I}}{\longmapsto} a \otimes \overline{r} \overset{\cong}{\longmapsto} a\overline{r} = \overline{ar} = 0$$

so indeed $a\cdot$ is the zero map. This means that:

$$\begin{array}{rcl} \mathrm{Tor}_0^R(R/I, R/I) & = & R/I, \\ \mathrm{Tor}_1^R(R/I, R/I) & = & \mathrm{Ker}(a\cdot)/\mathrm{Im}(0) \cong \mathrm{Ker}(a\cdot) = R/I, \\ \mathrm{Tor}_n^R(R/I, R/I) & = & 0 \text{ for } n \geq 2. \end{array}$$

To obtain Ext, we apply $\mathrm{Hom}_R(-, R/I)$ and truncate the complex, obtaining:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(R, R/I) & \xrightarrow{(a\cdot)^*} & \mathrm{Hom}_R(R, R/I) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & R/I & \xrightarrow[0]{a\cdot} & R/I & \longrightarrow & 0 \end{array}$$

where we use the isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_R(R, B) & \longrightarrow & B \\ \gamma & \longmapsto & \gamma(1) \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \longrightarrow & \mathrm{Hom}_R(R, B) \\ b & \longmapsto & \gamma_b \end{array} \quad \text{where} \quad \begin{array}{ccccc} \gamma_b & : & R & \longrightarrow & B \\ & & r & \longmapsto & rb \end{array}$$

and hence:

$$\overline{r} \overset{\cong}{\longmapsto} \gamma_{\overline{r}} \overset{(a\cdot)^*}{\longmapsto} a\gamma_{\overline{r}} \overset{\cong}{\longmapsto} a\overline{r} = \overline{ar} = 0$$

so indeed $a\cdot$ is the zero map. This means that:

$$\begin{array}{rcl} \mathrm{Ext}_R^0(R/I, R/I) & = & R/I, \\ \mathrm{Ext}_R^1(R/I, R/I) & = & \mathrm{Ker}(a\cdot)/\mathrm{Im}(0) \cong \mathrm{Ker}(a\cdot) = R/I, \\ \mathrm{Tor}_R^1(R/I, R/I) & = & 0 \text{ for } n \geq 2. \end{array}$$

Exercise 6

Let k a field and $R = k[x]/(x^2)$. Consider k to be an R-module on which x acts as multiplication by zero.

Note that k has a projective resolution:

$$\cdots \longrightarrow R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{\epsilon} k \longrightarrow 0$$

where $\epsilon(c) = c$ for $c \in k$ and $\epsilon(x) = 0$. We have $\mathrm{Ker}(\epsilon) = (x) = \mathrm{Im}(x\cdot)$, so the exactness follows. To obtain Tor, we apply $- \otimes_R k$ and truncate the complex, obtaining:

$$\begin{array}{ccccccc} \cdots \longrightarrow & R \otimes_R k & \xrightarrow{(x\cdot)\otimes 1_k} & R \otimes_R k & \longrightarrow 0 \\ & \downarrow \cong & & \downarrow \cong & \\ \cdots \longrightarrow & k & \underset{0}{\xrightarrow{x\cdot}} & k & \longrightarrow 0 \end{array}$$

where again we use the aforementioned isomorphism and hence:

$$c \overset{\cong}{\longmapsto} 1 \otimes c \overset{(x\cdot)\otimes 1_k}{\longmapsto} x \otimes c \overset{\cong}{\longmapsto} xc = 0$$

so indeed $x\cdot$ is the zero map. This means that:

$$\mathrm{Tor}_R^n(k,k) = \mathrm{Ker}(0)/\mathrm{Im}(0) = k \text{ for } n \geq 0,$$

the desired result.

Exercise 7

Let k a field and $R = k[x]/(x^r)$ for $r \geq 2$. Consider k to be an R-module on which x acts as multiplication by zero.

Note that k has a projective resolution:

$$\cdots \longrightarrow R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{\epsilon} k \longrightarrow 0$$

where $\epsilon(c) = c$ for $c \in k$ and $\epsilon(x) = 0$. We have $\mathrm{Ker}(\epsilon) = (x) = \mathrm{Im}(x\cdot)$, so the exactness follows. To obtain Ext, we apply $\mathrm{Hom}_R(-, k)$ and truncate the complex, obtaining:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Hom}_R(R,k) & \xrightarrow{(x\cdot)^*} & \mathrm{Hom}_R(R,k) & \longrightarrow 0 \\ & \downarrow \cong & & \downarrow \cong & \\ \cdots \longrightarrow & k & \xrightarrow[0]{x\cdot} & k & \longrightarrow 0 \end{array}$$

where we use the isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_R(R,B) & \longrightarrow & B \\ \gamma & \longmapsto & \gamma(1) \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \longrightarrow & \mathrm{Hom}_R(R,B) \\ b & \longmapsto & \gamma_b \end{array} \quad \text{where} \quad \begin{array}{cccc} \gamma_b : & R & \longrightarrow & B \\ & r & \longmapsto & rb \end{array}$$

and hence:

$$c \overset{\cong}{\longmapsto} \gamma_c \overset{(x\cdot)^*}{\longmapsto} x\gamma_c \overset{\cong}{\longmapsto} xc = 0$$

so indeed $a\cdot$ is the zero map. This means that:

$$\mathrm{Ext}_n^R(k,k) = \mathrm{Ker}(0)/\mathrm{Im}(0) = k \text{ for } n \geq 0,$$

the desired result.

Exercise 8

Let k be a field and $q \in k^{\times}$. Let $R = k_q[x, y]$ the quantum plane (we have the relation $yx = qxy$). Consider k to be an R-module on which x and y each act as zero.

1. We have the following free resolution of k as an R-module:

$$0 \longrightarrow R \xrightarrow{\alpha} R \oplus R \xrightarrow{\beta} R \xrightarrow{\epsilon} k \longrightarrow 0$$

where $\alpha = (qy, -x)^T$ and $\beta = (x, y)$. Clearly R and $R \oplus R$ are both free, hence projective. Moreover, the above is indeed exact:

(a) α injective: given two general elements in R, say $b = \sum_{i,j} b_{ij} x^i y^j$ and $c = \sum_{i,j} c_{ij} x^i y^j$ with $\alpha(b) = \alpha(c)$, this means:

$$\left(q \sum_{i,j} b_{ij} x^i y^{j+1}, -\sum_{i,j} b_{ij} x^{i+1} y^j \right) = \left(q \sum_{i,j} c_{ij} x^i y^{j+1}, -\sum_{i,j} c_{ij} x^{i+1} y^j \right)$$

so imposing equality term by term and then monomial by monomial, we obtain $b_{ij} = c_{ij}$, meaning that $b = c$, as desired.

(b) $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$: we first note that:

$$\begin{aligned} \mathrm{Im}(\alpha) &= \{r_1 \oplus r_2 \in R \oplus R : \exists r \in R, \alpha(r) = r_1 \oplus r_2\} \\ &= \{r_1 \oplus r_2 \in R \oplus R : \exists r \in R, qyr \oplus -xr = r_1 \oplus r_2\} \\ &= \{r_1 \oplus r_2 \in R \oplus R : \exists r \in R, r_1 = qyr \text{ and } r_2 = -xr\}, \end{aligned}$$

and:

$$\begin{aligned} \mathrm{Ker}(\alpha) &= \{r_1 \oplus r_2 \in R \oplus R : \beta(r_1 \oplus r_2) = 0\} \\ &= \{r_1 \oplus r_2 \in R \oplus R : xr_1 + yr_2 = 0\}. \end{aligned}$$

We clearly have $\mathrm{Im}(\alpha) \subset \mathrm{Ker}(\beta)$ since $xqyr + y(-xr) = qxyr - qxyr = 0$ by using the almost-commutativity. For the other inclusion, we note that if $r_1 \oplus r_2 \in \mathrm{Ker}(\beta)$ has r_1 with monomials $r_1^{i0} x^i$ with $r_1^{i0} \neq 0$ (that is, we cannot factor out a y), then the sum $xr_1 + yr_2$ will never be zero since yr_2 has all monomials with terms y, but xr_1 has monomials with terms only x, and this is a contradiction. Analogously, if $r_1 \oplus r_2 \in \mathrm{Ker}(\beta)$ has r_2 with monomials $r_2^{0j} y^j$ with $r_2^{0j} \neq 0$ (that is, we cannot factor out a x), then the sum $xr_1 + yr_2$ will never be zero since xr_1 has all monomials with terms x, but yr_2 has monomials with terms only y, and this is a contradiction. Hence given $r_1 \oplus r_2 \in \mathrm{Ker}(\beta)$ we can rewrite $r_2 = -xr$ for some $r \in R$, and now using this definition of r we note that since $xr_1 + yr_2 = 0$, then $xr_1 - yxr = 0$ so $r_1 = qyr$ by comparing the terms having x. This means $\alpha(r) = qyr \oplus -xr = r_1 \oplus r_2$ and $\mathrm{Ker}(\beta) \subset \mathrm{Im}(\alpha)$.

(c) $\mathrm{Im}(\beta) = \mathrm{Ker}(\epsilon)$: since ϵ is defined as $\epsilon(x) = 0$, $\epsilon(y) = 0$ and $\epsilon(c) = c$ for $c \in k$, this means:

$$\mathrm{Ker}(\epsilon) = (x, y)$$

the ideal generated by x and y. Moreover:

$$\mathrm{Im}(\beta) = \{r \in R : \exists r_1 \oplus r_2, r = xr_1 + yr_2\} = (x, y)$$

so we obtain what we wanted.

(d) ϵ surjective: given $c \in k \subset R$, by definition $\epsilon(c) = c$.

2. To find $\mathrm{Ext}^n_R(k, k)$ we apply $\mathrm{Hom}_R(-, k)$ and truncate the complex, obtaining:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(R, k) & \xrightarrow{\beta^*} & \mathrm{Hom}_R(R \oplus R, k) & \xrightarrow{\alpha^*} & \mathrm{Hom}_R(R, k) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & k & \xrightarrow{0} & k \oplus k & \xrightarrow{0} & k & \longrightarrow & 0
\end{array}$$

where we use the isomorphism:

$$\begin{array}{rcl} \mathrm{Hom}_R(R, B) & \longrightarrow & B \\ \gamma & \longmapsto & \gamma(1) \end{array} \quad \text{and} \quad \begin{array}{rcl} B & \longrightarrow & \mathrm{Hom}_R(R, B) \\ b & \longmapsto & \gamma_b \end{array} \quad \text{where} \quad \begin{array}{rcccl} \gamma_b & : & R & \longrightarrow & B \\ & & r & \longmapsto & rb \end{array}$$

and hence for $c, c_1, c_2 \in k$ and $r, r_1, r_2 \in R$ we have:

$$\begin{array}{rcl}
\beta^*(\gamma_c)(r_1 \oplus r_2) & = & \gamma_c(\beta(r_1 \oplus r_2)) = \gamma_c(xr_1 + yr_2) = xr_1c + yr_2c = 0 \\
\alpha^*(\gamma_{c1 \oplus c2})(r) & = & \gamma_{c_1 \oplus c_2}(\alpha(r)) = \gamma_{c_1 \oplus c_2}(qyr, -xr)^T = qyrc_1 + (-xrc_2) = 0
\end{array}$$

so indeed α^* and β^* are both the zero map. This means that:

$$\begin{array}{rcl}
\mathrm{Ext}^0_R(k, k) & = & \mathrm{Ker}(0)/\mathrm{Im}(0) \cong k \\
\mathrm{Ext}^1_R(k, k) & = & \mathrm{Ker}(0)/\mathrm{Im}(0) \cong k \oplus k, \\
\mathrm{Ext}^2_R(k, k) & = & \mathrm{Ker}(0)/\mathrm{Im}(0) \cong k \\
\mathrm{Tor}^1_R(k, k) & = & 0 \text{ for } n \geq 3.
\end{array}$$

Exercise 9

We consider the extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$:

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\alpha} \mathbb{Z}_4 \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0$$

with α and β the canonical inclusion and projection. To find the element of $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$ corresponding to this extension, we follow the proof of the Theorem stating that there is a one to one correspondence between the equivalence classes of A by B and $\operatorname{Ext}^1_R(A, B)$. We thus first consider the projective resolution of $\mathbb{Z}_2$ given by:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0$$

and comparing them, we get the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \cong} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{\alpha} & \mathbb{Z}_4 & \xrightarrow{\beta} & \mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

where τ is the canonical projection and the dotted arrows are our guesses for the arrows guaranteed by the Comparison Lemma, and they should make the diagram commute. They indeed work since:

$$\begin{aligned}
\pi 0 &= 00 \\
\tau 2 &= \alpha \pi \\
\pi &= \beta \tau
\end{aligned}$$

where the first equation needs no further justification, the second holds because the natural inclusion of α is basically multiplying by 2 an element of $\mathbb{Z}_2$ (because the group structure needs to be preserved) and the third holds because projecting directly or projecting through a ring containing $\mathbb{Z}_2$ is exactly the same (this holds in general, not only here).

Thus by the proof, the $\mathbb{Z}$-homomorphism $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}_2$ on the second column from the left induces (taking classes) an element in $\operatorname{Ext}^1_R(\mathbb{Z}_2, \mathbb{Z}_2)$ which is the one corresponding to this extension. By the standard route we can easily see that $\operatorname{Ext}^1_R(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2/2\mathbb{Z}_2 \cong \mathbb{Z}_2$, and since $\pi(1) = 1$ the map π is not the zero map, which is the identity additive element, so π corresponds to the non identity element of $\operatorname{Ext}^1_R(\mathbb{Z}_2, \mathbb{Z}_2)$.

References

[1] T. W. Hungerford, *Algebra*, Springer-Verlag, 2000.