Introduction to Commutative and Homological Algebra - Homework 2

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Exercise 1

Let R a ring, A, B submodules of M a left R module and C a left R module. We notice that the natural projection:

$$\begin{array}{rcccl} \pi & : & A \oplus B & \longrightarrow & A + B \\ & & (a, b) & \longmapsto & a + b \end{array}$$

is clearly surjective and induces a short exact sequence:

$$0 \longrightarrow \ker(\pi) \longrightarrow A \oplus B \longrightarrow A + B \longrightarrow 0.$$

Moreover:

$$\begin{aligned} \ker(\pi) &= \{(a, b) \in A \oplus B : 0 = \pi(a, b) = a + b\} = \{a \in A, b \in B : b = -a\} \\ &= \{a \in A : -a \in B\} = \{a \in A : a \in B\} = A \cap B, \end{aligned}$$

and in fact this is not only an equality of sets but an isomorphism of left R modules via:

$$\begin{array}{rcccl} \phi & : & \ker(\pi) & \longrightarrow & A \cap B \\ & & (a, -a) & \longmapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcccl} \psi & : & A \cap B & \longrightarrow & \ker(\pi) \\ & & a & \longmapsto & (a, -a) \end{array}$$

since both ϕ and ψ are R homomorphisms and inverses of each other; for every $a, a' \in A$ and $r \in R$ we have:

$$\begin{aligned} \phi((a, -a) + (a' - a')) &= \phi(a + a', -a - a') = a + a' = \phi(a, -a) + \phi(a', -a') \\ \phi(r(a, -a)) &= \phi(ra, -ra) = ra = r\phi(a, -a) \\ \psi(a + a') &= (a + a', -a - a') = (a, -a) + (a' - a') = \psi(a) + \psi(a') \\ \psi(ra) &= (ra, -ra) = r(a, -a) = r\psi(a) \\ \psi \circ \phi(a, -a) &= \psi(a) = (a, -a) \\ \phi \circ \psi(a) &= \phi(a, -a) = a. \end{aligned}$$

Thus the short exact sequence can be rewritten as:

$$0 \longrightarrow A \cap B \longrightarrow A \oplus B \longrightarrow A + B \longrightarrow 0,$$

and applying the Second Long Exact Sequence for Ext we obtain:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(A + B, C) \longrightarrow \mathrm{Hom}_R(A, C) \oplus \mathrm{Hom}_R(B, C) \longrightarrow \mathrm{Hom}_R(A \cap B, C) \longrightarrow \\ \mathrm{Ext}^1_R(A + B, C) \longrightarrow \mathrm{Ext}^1_R(A, C) \oplus \mathrm{Ext}^1_R(B, C) \longrightarrow \mathrm{Ext}^1_R(A \cap B, C) \longrightarrow \cdots \end{aligned}$$

where we used that $\mathrm{Ext}^n_R(A \oplus B, C) \cong \mathrm{Ext}^n_R(A, C) \oplus \mathrm{Ext}^n_R(B, C)$ for all $n \geq 0$ as proven in Homework 1. This is what we wanted to prove.

Exercise 2

Let $S \subset R$ rings, B and S module and A an R module, also denoting A its restriction as S module, and $R \otimes_S B$ the induction of B to an R module.

1. Consider the map:

$$\begin{array}{rcccc} \psi & : & \mathrm{Hom}_S(B,A) & \longrightarrow & \mathrm{Hom}_R(R\otimes_S B, A) \\ & & f & \longmapsto & \psi(f) \end{array} , \quad \begin{array}{rcccc} \psi(f) & : & R\otimes_S B & \longrightarrow & A \\ & & r\otimes b & \longmapsto & rf(b) \end{array}$$

where $\psi(f)$ is induced by the map:

$$\begin{array}{rcccc} \tilde{f} & : & R\times B & \longrightarrow & A \\ & & (r,b) & \longmapsto & rf(b) \end{array}$$

which is indeed S-balanced since for every $r, r' \in R$, $b, b' \in B$, $s \in S$ we have:

$$\begin{aligned} \tilde{f}(r, a+a') &= r(a+a') = ra + ra' = \tilde{f}(r,a) + \tilde{f}(r,a') \\ \tilde{f}(r+r', a) &= (r+r')a = ra + r'a = \tilde{f}(r,a) + \tilde{f}(r',a) \\ \tilde{f}(rs, a) &= (rs)a = r(sa) = \tilde{f}(r, sa) \end{aligned}$$

where we for the last equation we used that since $S \subset R$ their action on B is associative. Moreover, $\psi(f)$ is indeed an R homomorphism since for every $r, r' \in R$, $b, b' \in B$ we have:

$$\begin{aligned} \psi(f)(r\otimes b + r'\otimes b') &= \psi(f)(r\otimes b) + \psi(f)(r'\otimes b') \\ \psi(f)(r'(r\otimes b)) &= \psi(f)((r'r)\otimes b) = (r'r)f(b) = r'(rf(b)) = r'\psi(f)(r\otimes b) \end{aligned}$$

again using that the action is associative, and thus ψ is well defined. Clearly ψ is a group homomorphism; for every $f, g \in \mathrm{Hom}_S(B,A)$, $r \in R$, $b \in B$ we have:

$$\psi(f+g)(r\otimes b) = r(f+g)(b) = rf(b) + rg(b) = \psi(f)(r\otimes b) + \psi(g)(r\otimes b).$$

Considering now the map:

$$\begin{array}{rcccc} \phi & : & \mathrm{Hom}_R(R\otimes_S B, A) & \longrightarrow & \mathrm{Hom}_S(B,A) \\ & & f & \longmapsto & \phi(f) \end{array} , \quad \begin{array}{rcccc} \phi(f) & : & B & \longrightarrow & A \\ & & b & \longmapsto & f(1\otimes b) \end{array}$$

where $\phi(f)$ is a S homomorphism since for every $b, b' \in B$, $s \in S$ we have:

$$\begin{aligned} \phi(f)(b+b') &= f(1\otimes(b+b')) = f(1\otimes b + 1\otimes b') = f(1\otimes b) + f(1\otimes b') \\ &= \phi(f)(b) + \phi(f)(b') \\ \phi(f)(sb) &= f(1\otimes(sb)) = f(s\otimes b) = f(s(1\otimes b)) = sf(1\otimes b) = s\phi(f)(b) \end{aligned}$$

where for the last equation we used the module structure of $R\otimes_S B$ and that f is an R module homomorphism. Clearly ϕ is a group homomorphism; for every $f, g \in \mathrm{Hom}_R(R\otimes_S B, A)$, $b \in B$ we have:

$$\phi(f+g)(b) = (f+g)(1\otimes b) = f(1\otimes b) + g(1\otimes b) = \phi(f)(b) + \phi(g)(b).$$

Finally, ψ and ϕ are inverses of each other; for every $f \in \mathrm{Hom}_S(B, A)$, $g \in \mathrm{Hom}_R(R \otimes_S B, A)$, $r \in R$, $b \in B$ we have:

$$\begin{aligned} \phi \circ \psi(f)(b) &= \phi(\psi(f))(b) = \psi(f)(1 \otimes b) = f(b) \\ \psi \circ \phi(g)(r \otimes b) &= \psi(\phi(g))(r \otimes b) = r\phi(g)(b) = rg(1 \otimes b) = g(r(1 \otimes b)) = g(r \otimes b), \end{aligned}$$

and thus $\mathrm{Hom}_S(B, A) \cong \mathrm{Hom}_R(R \otimes_S B, A)$ as desired.

2. Consider R a projective right S module under multiplication. Given $P.$ a projective resolution of B over S, we have an exact sequence:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} B \longrightarrow 0,$$

inducing an exact sequence:

$$\cdots \longrightarrow R \otimes_S P_1 \longrightarrow R \otimes_S P_0 \xrightarrow{\mathrm{id}_R \otimes \epsilon} B \longrightarrow 0,$$

because the functor $R \otimes_S -$ is right exact. Notice that this is a projective resolution of $R \otimes_S B$ over R: for every $n \geq 0$ we have that P_n is projective over S, meaning that it is a direct summand of a free S module, say $F_S = P_n \oplus Q_n$. Now $R \cong R \otimes_S F_S \cong (R \otimes_S P_n) \oplus (R \otimes_S Q_n)$ and thus $R \otimes_S P_n$ is a direct summand of a free R module, hence it is projective as R module.

Applying now $\mathrm{Hom}_S(-, A)$ to the first resolution and $\mathrm{Hom}_R(-, A)$ to the second resolution we obtain the truncated complexes:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_S(P_0, A) \longrightarrow \mathrm{Hom}_S(P_1, A) \longrightarrow \cdots \\ 0 &\longrightarrow \mathrm{Hom}_R(R \otimes_S P_0, A) \longrightarrow \mathrm{Hom}_R(R \otimes_S P_1, A) \longrightarrow \cdots \end{aligned}$$

which are isomorphic in virtue of what we proven in the previous section, so in particular they give the same homology. The homology of the first complex is $\mathrm{Ext}^n_S(B, A)$, and the homology of the second complex is $\mathrm{Ext}^n_R(R \otimes_S B, A)$, meaning that for every $n \geq 0$ we have $\mathrm{Ext}^n_S(B, A) \cong \mathrm{Ext}^n_R(R \otimes_S B, A)$, as desired.

Exercise 3

Consider $R = \mathbb{Z}_4$, $M = \mathbb{Z}_2$ as an R module via $M \cong R/(2)$, that is, if $R = \{0,1,2,3\}$ then $M = \{\overline{0}, \overline{1}\}$ with $\overline{0} = \{0,2\}$ and $\overline{1} = \{1,3\}$ and $\pi : R \longrightarrow M$ has $\pi(1) = \overline{1}$. We consider the R resolution of M:

$$\cdots \longrightarrow \mathbb{Z}_4 \xrightarrow{(2\cdot)} \mathbb{Z}_4 \xrightarrow{(2\cdot)} \mathbb{Z}_4 \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0$$

which is clearly R projective. Since π is injective by construction, we have $\ker(\pi) = \{0,2\}$ and since:

$$\begin{array}{rcccc} 2\cdot & : & \mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_4 \\ & & 0 & \longrightarrow & 0 \\ & & 1 & \longrightarrow & 2 \\ & & 2 & \longrightarrow & 0 \\ & & 3 & \longrightarrow & 2 \end{array}$$

we clearly have $\ker(2\cdot) = \{0,2\} = \operatorname{Im}(2\cdot)$ and thus the resolution is exact. Applying now $\operatorname{Hom}_R(-, M)$ we obtain the truncated complex:

$$0 \longrightarrow \operatorname{Hom}_R(R, M) \xrightarrow{(2\cdot)^*} \operatorname{Hom}_R(R, M) \xrightarrow{(2\cdot)^*} \operatorname{Hom}_R(R, M) \xrightarrow{(2\cdot)^*} \cdots$$

and since for every $r \in R$ and $f \in \operatorname{Hom}_R(R, M)$ we have:

$$(2\cdot)^*(f)(r) = f(2 \cdot r) = f(0) = 0$$

this means that $(2\cdot)^* \equiv 0$ as maps. Hence:

$$\operatorname{Ext}^n_R(M, M) = \ker(0)/\operatorname{Im}(0) = \ker(0) = \operatorname{Hom}_R(R, M) \cong \mathbb{Z}_2 \text{ for } n \geq 1$$

since a function $f \in \operatorname{Hom}_R(M, M)$ is determined by the image of 1, and we have two choices. Similarly, we know that $\operatorname{Ext}^0_R(M, M) \cong \operatorname{Hom}_R(M, M) \cong \mathbb{Z}_2$. This means that we have non zero Ext in every degree.

Knowing this, suppose that $\operatorname{pd}_R(M) = k < \infty$, then there is a R projective resolution of M of the form:

$$0 \longrightarrow Q_k \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

which applying $\operatorname{Hom}_R(-, M)$ induces a truncated complex:

$$0 \longrightarrow \operatorname{Hom}_R(Q_0, M) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_R(Q_k, M) \longrightarrow 0,$$

meaning that for $n \geq k+1$ we have $\operatorname{Ext}^n_R(M, M) = 0$, a contradiction. Thus we must have $\operatorname{pd}_R(M) = \infty$, as desired.

Exercise 4

1. Let A a right R module and B a left R module. Suppose that $\mathrm{pd}_R(A) = m$, then there is a R projective resolution of A:

   $$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

   which applying $- \otimes_R B$ induces the truncated complex:

   $$0 \longrightarrow P_m \otimes_R B \longrightarrow \cdots \longrightarrow P_0 \otimes_R B \longrightarrow 0,$$

   and thus taking homology this means that $\mathrm{Tor}_n^R(A, B) = 0$ for $n \geq m + 1$.

   For the following reasoning, we consider the equivalent definition of Tor that allows us to take a projective resolution of the right component, tensor with the left and take homology. Suppose now that $\mathrm{pd}_R(B) = m$, then there is a R projective resolution of B:

   $$0 \longrightarrow Q_m \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow B \longrightarrow 0,$$

   which applying $A \otimes_R -$ induces the truncated complex:

   $$0 \longrightarrow A \otimes_R Q_m \longrightarrow \cdots \longrightarrow A \otimes_R Q_0 \longrightarrow 0,$$

   and thus taking homology this means that $\mathrm{Tor}_n^R(A, B) = 0$ for $n \geq m + 1$.

2. Suppose that $\mathrm{gldim}(R) = \infty$, then clearly $\mathrm{Tordim}(R) \leq \infty = \mathrm{gldim}(R)$ always holds. Suppose that $\mathrm{gldim}(R) = n < \infty$, that is, there exists a left R module V with $\mathrm{pd}_R(V) = n$ and there are no other left R modules with projective dimension greater than n. Thus for every couple of left R modules A, B we have that $\mathrm{pd}_R(A), \mathrm{pd}_R(B) \leq n$, meaning that $\mathrm{Tor}_k^R(A, B) = 0$ for $k \geq n+1$ as proven in the previous section, and thus to have $\mathrm{Tor}_k^R(A, B) \neq 0$ we need $k \leq n$. Since this is true for every pair of left R modules A, B, we obtain that $\mathrm{Tordim}(R) \leq n = \mathrm{gldim}(R)$.

Exercise 5

Let R a ring and A, B be R modules. Suppose first that $\mathrm{pd}_R(A) = n < \infty$. Consider the two exact sequences:

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$
$$0 \longrightarrow B \longrightarrow A \oplus B \longrightarrow A \longrightarrow 0$$

where the maps are the natural component-wise inclusions and projections making the sequences exact. We proved in class that under the circumstances of the first short exact sequence we have $\mathrm{pd}_R(A \oplus B) \leq \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(B)\}$. If $\mathrm{pd}_R(A \oplus B) < \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(B)\}$, then in virtue of what we proved in class we must have $\mathrm{pd}_R(B) = \mathrm{pd}_R(A) + 1$. Using this same result for the second short exact sequence, we still have a strict inequality $\mathrm{pd}_R(A \oplus B) < \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(B)\}$ meaning that $\mathrm{pd}_R(A) = \mathrm{pd}_R(B) + 1$. Thus $n = \mathrm{pd}_R(A) = \mathrm{pd}_R(B) + 1 = \mathrm{pd}_R(A) + 2 = n + 2$, a contradiction (this is regardless of $\mathrm{pd}_R(B)$), so we indeed must have $\mathrm{pd}_R(A \oplus B) = \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(B)\}$. This exact same argument proves that if $\mathrm{pd}_R(B) = n < \infty$ then $\mathrm{pd}_R(A \oplus B) = \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(B)\}$.

Suppose now that $\mathrm{pd}_R(A) = \infty = \mathrm{pd}_R(B)$. Suppose that $\mathrm{pd}_R(A \oplus B) = n \leq \infty$. We proved in class that this happens if and only if $\mathrm{Ext}_R^k(A \oplus B, C) = 0$ for all $k > n$. By the Homework 1, we have:

$$\mathrm{Ext}_R^k(A, C) \oplus \mathrm{Ext}_R^k(B, C) \cong \mathrm{Ext}_R^k(A \oplus B, C) = 0 \text{ for all } k > n,$$

that is $\mathrm{Ext}_R^k(A, C) = 0 = \mathrm{Ext}_R^k(B, C)$. Again, this happens if and only if $\mathrm{pd}_R(A), \mathrm{pd}_R(B) \leq n < \infty$, a contradiction. We thus have $\mathrm{pd}_R(A \oplus B) = \infty = \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(B)\}$, as desired.

References

[1] T. W. Hungerford, *Algebra*, Springer-Verlag, 2000.