Introduction to Commutative and Homological Algebra - Homework 3

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November 3rd, 2017

Exercise 1

Let $I, J \subseteq A$ ideals of a ring. We prove that $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. For this, it is enough to prove three inclusions:

$\sqrt{IJ} \subseteq \sqrt{I \cap J}$: Let $a \in \sqrt{IJ}$, then $a^n \in IJ$ for some $n \in \mathbb{N}$, meaning that we can write $a^n = \sum_{i=1}^k b_i c_i$ for some $b_i \in I$, $c_i \in J$, $1 \leq i \leq k$. Notice that since both I and J are ideals, then $b_i c_i \in I$ and $b_i c_i \in J$ for every $1 \leq i \leq k$, meaning that $b_i c_i \in I \cap J$ for $1 \leq i \leq k$, and since the intersection of ideals is an ideal, $a^n = \sum_{i=1}^k b_i c_i \in I \cap J$. This means $a \in \sqrt{I \cap J}$.

$\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$: Let $a \in \sqrt{I \cap J}$, then $a^n \in I \cap J$ for some $n \in \mathbb{N}$, meaning $a^n \in I$ and $a^n \in J$. Thus $a \in \sqrt{I}$ and $a \in \sqrt{J}$, so $a \in \sqrt{I} \cap \sqrt{J}$.

$\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{IJ}$: Let $a \in \sqrt{I} \cap \sqrt{J}$, then $a \in \sqrt{I}$ and $a \in \sqrt{J}$, so $a^n \in I$ and $a^m \in J$. for some $n, m \in \mathbb{N}$. Then $a^{n+m} = a^n a^m \in IJ$, thus $a \in \sqrt{IJ}$.

So we have $\sqrt{IJ} \subseteq \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{IJ}$, meaning that these inclusions must be equalities, as desired.

Exercise 2

Let $\phi : A \longrightarrow B$ be a ring homomorphism, $I \subseteq B$ and ideal. Then:

$\phi^{-1}\left(\sqrt{I}\right) \subseteq \sqrt{\phi^{-1}(I)}$: Let $a \in \phi^{-1}\left(\sqrt{I}\right)$, then $\phi(a) \in \sqrt{I}$. This means that $\phi(a)^n \in I$ for some $n \in \mathbb{N}$, and thus:

$$\phi(a^n) = \phi(a \overset{n}{\cdots} a) = \phi(a) \overset{n}{\cdots} \phi(a) = \phi(a)^n \in I$$

using that ϕ is a ring homomorphism. Then $a^n \in \phi^{-1}(I)$, so $a \in \sqrt{\phi^{-1}(I)}$.

$\sqrt{\phi^{-1}(I)} \subseteq \phi^{-1}\left(\sqrt{I}\right)$: Let $a \in \sqrt{\phi^{-1}(I)}$, meaning that $a^n \in \phi^{-1}(I)$ for some $n \in \mathbb{N}$. Thus $\phi(a^n) \in I$, and now:

$$\phi(a)^n = \phi(a) \overset{n}{\cdots} \phi(a) = \phi(a \overset{n}{\cdots} a) = \phi(a^n) \in I$$

using that ϕ is a ring homomorphism. Then $\phi(a) \in \sqrt{I}$, so $a \in \phi^{-1}\left(\sqrt{I}\right)$.

So we have the desired equality $\phi^{-1}\left(\sqrt{I}\right) = \sqrt{\phi^{-1}(I)}$.

Exercise 3

Let $n \in \mathbb{N}$, $n \geq 2$. We find $\mathrm{nil}(\mathbb{Z}_n)$. For this, we will use the known fact that the prime ideals of $\mathbb{Z}$ are the ones of the form $\mathbb{Z}_p$ for $p \in \mathbb{Z}$ prime. To translate this information to $\mathbb{Z}_n$, we use several standard facts: given R is a commutative ring and $I \subseteq R$ an ideal, then there is a one to one correspondence between the set of ideals of R containing I and the set of ideals of R/I (and this correspondence is given by taking quotient by I) [1, Theorem 2.13 (p. 126)], and an ideal $P \subseteq R$ is prime if and only if R/P is an integral domain. Using these two facts and the Third Isomorphism Theorem for rings, it is immediate to see that all the prime ideals of R/I are of the form J/I for some prime ideal $J \subseteq R$ with $I \subseteq J$.

We know that a prime ideal $\mathbb{Z}_p$ contains $\mathbb{Z}_n$ if and only if p divides n, and applying the above, we obtain that all the prime ideals of $\mathbb{Z}_n$ are of the form $\mathbb{Z}_p/\mathbb{Z}_n$ for $p|n$. Thus:

$$\mathrm{nil}(\mathbb{Z}_n) = \bigcap_{\substack{p\in\mathbb{Z} \text{ prime} \\ p|n}} \mathbb{Z}_p/\mathbb{Z}_n$$

is the desired result.

Exercise 4

Let $\phi : A \longrightarrow B$ be a surjective ring homomorphism.

1. Prove that $\phi(\mathrm{rad}(A)) \subseteq \mathrm{rad}(B)$. For this, we use that the Jacobson radical of a ring is the intersection of all maximal ideals of that ring. Notice:

$$\begin{aligned} a \in \bigcap_{\substack{M \subseteq A \text{ ideal} \\ M \text{ maximal}}} M &\implies a \in M \quad \forall M \subseteq A \text{ ideal, } M \text{ maximal} \\ &\implies \phi(a) \in \phi(M) \quad \forall M \subseteq A \text{ ideal, } M \text{ maximal} \\ &\implies \phi(a) \in \bigcap_{\substack{M \subseteq A \text{ ideal} \\ M \text{ maximal}}} \phi(M) \end{aligned}$$

so:

$$\phi\left(\bigcap_{\substack{M \subseteq A \text{ ideal} \\ M \text{ maximal}}} M \right) \subseteq \bigcap_{\substack{M \subseteq A \text{ ideal} \\ M \text{ maximal}}} \phi(M) \subseteq \bigcap_{\substack{N \subseteq B \text{ ideal} \\ N \text{ maximal}}} N$$

where for the last inclusion we use that the surjective homomorphic image of an ideal is an ideal, meaning that it is contained in a maximal one. Since every maximal ideal $N \subseteq B$ has as preimage $\phi^{-1}(N)$ a maximal ideal again by surjectivity of ϕ, this guarantees that every maximal ideal of B is present in the last intersection. Thus $\phi(\mathrm{rad}(A)) \subseteq \mathrm{rad}(B)$.

The above was my first approach to the problem. I believe it to be correct, but I noticed that the very last idea of why all maximal ideals have to be contained in the intersection of the images can be used to much greater effect, as displayed below. Let $N \subseteq B$ be a maximal ideal. Then $\phi^{-1}(N) \subseteq A$ is a maximal ideal by surjectivity of ϕ (this is a standard fact, it is easily proved using the First Isomorphism Theorem for rings), meaning that $\mathrm{rad}(A) \subseteq \phi^{-1}(N)$ since the Jacobson radical is the intersection of all maximal ideals. This implies $\phi(\mathrm{rad}(A)) \subseteq N$ for every maximal ideal $N \subseteq B$, so indeed $\phi(\mathrm{rad}(A)) \subseteq \mathrm{rad}(B)$.

2. Give an example where the inclusion is strict. Consider the natural projection $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}_4$ (which is surjective), since both are principal ideal domains, being a maximal ideal is the same thing as being a prime ideal. We know that $\mathrm{rad}(\mathbb{Z}) = (0)$ since there are infinite odd prime numbers, and by the previous Exercise we know that the only prime ideal of $\mathbb{Z}_4$ is $\mathbb{Z}_2/\mathbb{Z}_4$. Since $2 \in \mathbb{Z}_2/\mathbb{Z}_4$ but $2 \notin (0)$ we have $\mathrm{rad}(\mathbb{Z}) = (0) \subsetneq \mathbb{Z}_2/\mathbb{Z}_4 = \mathrm{rad}(\mathbb{Z}_4)$, as desired.

Exercise 5

Let A be a local ring, we prove that A has no idempotent elements other than 1 and 0. Let M be the unique maximal ideal of A. Given $a \in A$ with $a^2 = a$, we have $a \in M$ or $a \notin M$.

If $a \notin M$, since M is the ideal of all non-units of A, then a is a unit. This means that there is an element $b \in A$ with $ba = 1$. Hence $a = 1a = baa = ba^2 = ba = 1$.

If $a \in M$, then $1 - a$ is a unit since $M = \text{rad}(A)$ and the elements x of the Jacobson radical are such that $1 + yx$ is a unit for every $y \in A$. This means that there is an element $c \in A$ with $(1-a)c = 1$. Hence $a = a1 = a(1-a)c = (a - a^2)c = 0c = 0$.

Hence if $a^2 = a$, then either $a = 1$ or $a = 0$, what we wanted to prove.

Exercise 6

Let A be a local ring with maximal ideal I, let M, N be finitely generated A modules. We prove that if $M \otimes_A N = 0$ then $M = 0$ or $N = 0$.

For this, we first notice that the result is true if A is a field: let k a field, M, N finitely generated k modules, say $M \cong k^m$ and $N \cong k^n$, with $M \neq 0 \neq N$, which is equivalent to $m \neq 0 \neq n$. Then $M \otimes_k N \cong k^m \otimes_k k^n \cong k \ncong 0$. Hence by contrapositive, if $M \otimes_k N = 0$, then either $m = 0$, implying $M = 0$, or $n = 0$, implying $N = 0$.

Now in the general case, assume $M \otimes_A N = 0$, now:

$$\begin{aligned} 0 &= M \otimes_A N \Longrightarrow \\ 0 &= (A/I) \otimes_A M \otimes_A N \cong ((A/I) \otimes_A M) \otimes_A N \cong (M/IM) \otimes_A N \end{aligned}$$

where we have used [2, Proposition 2.7 (p. 612)] for the last isomorphism. Notice how M/IM has a structure of A/I modules, and the isomorphism $(A/I) \otimes_A M \cong M/IM$ is in fact both an isomorphism of A modules and of A/I modules. Now:

$$\begin{aligned} 0 &= (M/IM) \otimes_A N \Longrightarrow \\ 0 &= (M/IM \otimes_{A/I} A/I) \otimes_A N \cong M/IM \otimes_{A/I} (A/I \otimes_A N) \cong M/IM \otimes_{A/I} N/IN \end{aligned}$$

where for the first isomorphism we have used [3, Ejercicio 2.15 (p. 31)][1] since A/I is both and A/I module and a A module, and both structures are compatible since the first is induced by the second, and for the last isomorphism we have again used [2, Proposition 2.7 (p. 612)] so N/IN is indeed an A/I module.

Since A/I is a field, this implies that either $M/IM = 0$ or $N/IN = 0$, so either $M = IM$ or $N = IN$. Since A is local, the maximal ring I is exactly the Jacobson radical (in particular it is contained in it), so by Nakayama's Lemma[3, Proposición 2.15 (p. 24)][2] we have that either $M = 0$ or $N = 0$, as desired.

[1] I apologize for referencing a non-English version of the book and an exercise instead of a Theorem or Proposition.

[2] I again apologize for referencing a non-English version of the book.

Exercise 7

Let A a ring. We want to prove that if $A^m \cong A^n$ for some non-zero $m, n \in \mathbb{N}$, then $m = n$. The name of this concept is the invariant basis number, where we say that a ring has the invariant basis number property if all its finitely generated free modules have a well defined rank.

We first prove that if $I \subseteq A$ is an ideal such that A/I has the invariant basis number property, then A also has the invariant basis number property. For this, suppose we have $A^m \cong A^n$ for some non-zero $m, n \in \mathbb{N}$, then:

$$(A/I)^m \cong (A/I) \otimes_A A^m \cong A^m/IA^m \cong A^n/IA^n \cong (A/I) \otimes_A A^n \cong (A/I)^n$$

where we use [2, Proposition 2.7 (p. 612)] in the second and fourth isomorphisms and the hypothesis in the third isomorphism. Since this is an isomorphism of A/I modules (having the invariant basis number property), then $m = n$, so A also has the invariant basis number property.

Notice now that since a finite dimensional vector space has unique dimension, a field always has the invariant basis number property. Taking now I to be a maximal ideal of A, which we know that always exist, then A/I is a field since A is commutative. Thus A/I has the invariant basis number property, so A has the invariant basis number property, which is what we wanted to prove.

References

[1] T. W. Hungerford, *Algebra*, Springer-Verlag, 2000.

[2] S. Lang, *Algebra (Revised 3rd Edition)*, Springer-Verlag, 2002.

[3] M. F. Atiyah, I. G. Macdonald *Introducción al Álgebra Conmutativa*, Reverté, 1978 (2015 reprint).