Introduction to Commutative and Homological Algebra - Homework 4

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Exercise 1

Let A be a ring with ideals $I_1, \ldots, I_n$ with $I_1 \cap \cdots I_n = (0)$. Prove that if A/I_j is noetherian for each $1 \leq j \leq n$, then A is noetherian.

Consider the map:

$$\begin{array}{rcccc} f & : & A & \longrightarrow & A/I_1 \oplus \cdots \oplus A/I_n \\ & & a & \longmapsto & (\overline{a}, \cdots, \overline{a}) \end{array}$$

which is clearly a ring homomorphism since it is product of projections. Moreover, it is injective since if $f(a) = (\overline{a}, \ldots, \overline{a}) = (\overline{0}, \ldots, \overline{0})$ then $\overline{a} = \overline{0}$ when seen in I_j for every $1 \leq j \leq n$, this means $a \in I_j$ for every $1 \leq j \leq n$. Hence $a \in I_1 \cap \cdots I_n = (0)$ so $a = 0$.

The above is seeing them as rings. However, given I an ideal of A we can see A/I as an A module by the induced multiplication $a\overline{m} = \overline{am}$ for all $a \in A$, $m \in A/I$ (the module properties hold because of the subring structure of A/I). Suppose further that A/I is noetherian as a ring. This means that if $J \subseteq A/I$ is an ideal, it is finitely generated over A/I, say $J \cong A/I\overline{g_1} \oplus \cdots \oplus A/I\overline{g_k}$ with $g_1, \ldots, g_k \in A$. Notice that $A/I \cong A\overline{1}$ as A modules, hence we obtain that $J \cong A\overline{1}\overline{g_1} \oplus \cdots \oplus A\overline{1}\overline{g_k} \cong A\overline{g_1} \oplus \cdots \oplus A\overline{g_k}$ as A modules, and thus J is finitely generated. Thus A/I is noetherian as A-module.

Since we are in the situation above, and the ring homomorphism f preserves the natural A module structure on $A/I_1 \oplus \cdots \oplus A/I_n$ given by point-wise multiplication (again, using the induced ring structure), we obtain the short exact sequence of A modules:

$$0 \longrightarrow A \longrightarrow A/I_1 \oplus \cdots \oplus A/I_n \longrightarrow \text{Coker}(f) \longrightarrow 0$$

where the middle one is noetherian. This means that the side ones are also noetherian as A modules. In particular, A is then noetherian as a ring, as desired.

Exercise 2

Let A, B, C be rings, let $\phi : A \longrightarrow C$, $\psi : B \longrightarrow C$ ring homomorphisms. Prove that if A, B are noetherian and ϕ, ψ are surjective then the fiber product $A \times_C B$ is noetherian.

Consider the diagram:

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_A} & A \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow[\psi]{} & C \end{array}$$

where $\pi_A : A \times_C B \longrightarrow A$ and $\pi_B : A \times_C B \longrightarrow B$ are the projections into the first and second component respectively. We have that $\mathrm{Ker}(\pi_A)$ and $\mathrm{Ker}(\pi_B)$ are ideals of $A \times_C B$, and $\mathrm{Ker}(\pi_A) \cap \mathrm{Ker}(\pi_B) = \{(a, b) \in A \times_C : a = \pi_A(a, b) = 0 = \pi_B(a, b) = b\} = \{(0, 0)\}$.

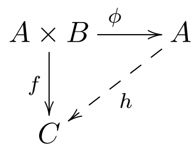
Let $a \in A$, then $\phi(a) \in C$, and since ψ is surjective we have that there exists $b \in B$ with $\psi(b) = \phi(a)$, meaning that $(a, b) \in A \times_C B$ and $\pi_A(a, b) = a$. Thus π_A is surjective, meaning that by the First Isomorphism Theorem $A \times_C B/\mathrm{Ker}(\pi_A) \cong \mathrm{Im}(\pi_A) = A$ is noetherian as a ring. Analogously, we obtain that π_B is surjective and $A \times_C B/\mathrm{Ker}(\pi_B) \cong \mathrm{Im}(\pi_B) = B$ is noetherian as a ring.

We are now satisfying the hypothesis of Exercise 1 above. Applying it, we obtain that $A \times_C B$ is noetherian, as desired.

Exercise 3

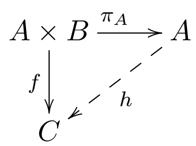
Let A, B be rings, $S = \{(1,1),(1,0)\}$ a multiplicative subset of $A \times B$. We show that $(A \times B)_S \cong A$.

To prove this, let C be another ring and $f : A \times B \longrightarrow C$ be any ring homomorphism with $f(s)$ invertible in C for every $s \in S$, we consider the diagram:



where we want to fix a ring homomorphism $\phi : A \times B \longrightarrow A$ and given f, find an unique ring homomorphism $h : A \longrightarrow C$ making the diagram commute.

Fixing $f : A \times B \longrightarrow C$ as in the diagram, we first notice that $(1,0)(a,b) - (1,1)(a,0) = 0$ for every $b \in B$, meaning that $f(a,0) = f((1,1)(a,0)) = f((1,0)(a,b)) = f(1,0)f(a,b)$ for every $b \in B$. Thus for every $b, b' \in B$ we have $f(1,0)f(a,b) = f(a,0) = f(1,0)f(a,b')$. Since $(1,0) \in S$ we have $f(1,0)$ invertible in C, so we multiply to the left by its inverse and obtain $f(a,b) = f(a,b')$ for every $b, b' \in B$. In particular, $f(1,1) = f(1,0)$. This allows us to define $\phi = \pi_A : A \times B \longrightarrow A$ the natural projection, that is, $\phi_A(a,b) = a$ for every $(a,b) \in A \times B$, which is clearly a ring homomorphism:



Define:

$$\begin{array}{rcccc} h & : & A & \longrightarrow & C \\ & & a & \longrightarrow & f(a,0) \end{array}$$

which is a ring homomorphism since f is a ring homomorphism. Notice how this makes the diagram commute since $f(a,b) = f(a,0) = h(a) = h(\pi_A(a,b))$ for every $(a,b) \in A \times B$. Moreover such an h is unique, since if we have $g : A \longrightarrow C$ such that $f = g\pi_A$ then $g(a) = g(\pi_A(a,0)) = f(a,0) = h(a)$ for every $a \in A$, meaning that $g = h$. Hence A equipped with $\pi_A : A \times B \longrightarrow A$ satisfies the Universal Property of the localization, meaning that $(A \times B)_S \cong A$, as desired.

Exercise 4

Let A a ring, S a multiplicative subset.

1. Let I an ideal of A, we show $\sqrt{I_S} = \left(\sqrt{I}\right)_S$.

   The equality as sets is easily seen. Given an element in $\left(\sqrt{I}\right)_S$, it is of the form a/s with $a \in \sqrt{I}$ and $s \in S$, that is, $a^n \in I$ for some $n \in \mathbb{N}$, so $(a/s)^n = a^n/s^n \in I_S$, meaning that $a/s \in \sqrt{I_S}$. This proves $\left(\sqrt{I}\right)_S \subseteq \sqrt{I_S}$. Given an element in $\sqrt{I_S}$, it is of the form a/s with $a^n/s^n = (a/s)^n \in I_S$ for some $n \in \mathbb{N}$, $a \in A$ and $s \in S$. This means that $a^n \in I$ and $a^n/s^n = b/t$ for some $b \in A$, $t \in S$ and hence there is $r \in S$ with $r(a^n t - bs^n) = 0$. In particular $(rat)^n \in I$ so $rat \in \sqrt{I}$, meaning that $a/s = rat/rst \in \left(\sqrt{I}\right)_S$. This proves $\sqrt{I_S} \subseteq \left(\sqrt{I}\right)_S$.

   Since the ring structure is the one inherited from A_S, they indeed are also equal when considered as rings.

2. We show $\mathrm{nil}(A_S) = \mathrm{nil}(A)A_S$.

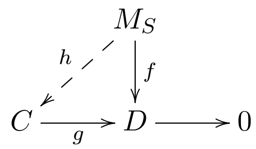
   The equality as sets is easily seen. Given an element in $\mathrm{nil}(A_S)$, it is of the form a/s with $a \in A$ and $s \in S$ with $a^n/s^n = (a/s)^n = 0/1$ for some $n \in \mathbb{N}$. This means that there is $t \in S$ with $ta^n = 0$, in particular $(ta)^n = 0$. Hence $a/s = at/st = (at)(1/st) \in \mathrm{nil}(A)A_S$. This proves $\mathrm{nil}(A_S) \subseteq \mathrm{nil}(A)A_S$. Given an element in $\mathrm{nil}(A)A_S$, it is of the form $b(a/s)$ with $b, a \in A$ and $s \in S$ with $b^n = 0$ for some $n \in \mathbb{N}$. Then $((ba)/s)^n = (b^n a^n)/s^n = 0$ so $b(a/s) = (ba)/s \in \mathrm{nil}(A_S)$. This proves $\mathrm{nil}(A)A_S \subseteq \mathrm{nil}(A_S)$.

   As before, the ring structure is the one inherited from A_S so they indeed are also equal when considered as rings.

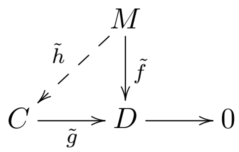
Exercise 5

Let A be a ring and S a multiplicative subset. Let M be a projective A module. Show that M_S is a projective A_S module.

Consider the diagram of A_S modules and A_S homomorphisms:



where we are given $g : C \longrightarrow D$ surjective, $f : M_S \longrightarrow D$ any morphism, we want to find $h : M_S \longrightarrow D$ a morphism making the diagram commute. For this, we first restrict all the modules to the induced action as A modules and all the given maps to the induced action as A homomorphisms. This yields the diagram of A modules and A homomorphisms:



where we identify M as the elements $m/1 \in M_S$, that is, $\tilde{f}(m) = f(m/1)$ for every $m \in M$. Since M is a projective A module, we obtain that there exists an A homomorphism $\tilde{h} : M \longrightarrow C$ making the diagram of A modules and A homomorphisms commute. We define now:

$$\begin{array}{rccc} h : & M_S & \longrightarrow & C \\ & h(m/s) & \longmapsto & s^{-1}\tilde{h}(m) \end{array}$$

which is well defined since $\tilde{h}(m) \in C$, we can multiply by s^{-1} in the structure of A_S module and if $m/s = n/t$ for some $m, n \in M$, $s, t \in S$, then there exists $r \in S$ with $r(mt - sn) = 0$. Applying $\tilde{h}$ we obtain $r(t\tilde{h}(m) - s\tilde{h}(n)) = 0$, meaning that $\tilde{h}(m)/s = \tilde{h}(n)/t$. This is an A_S homomorphism since:

$$\begin{aligned} h\left(\frac{m}{s} + \frac{m'}{s'}\right) &= h\left(\frac{ms' + sm'}{ss'}\right) = (ss')^{-1}\tilde{h}(ms' + sm') = (ss')^{-1}\tilde{h}(ms') \\ &+ (ss')^{-1}\tilde{h}(sm') = s^{-1}\tilde{h}(m) + s'^{-1}\tilde{h}(m') = h\left(\frac{m}{s}\right) + h\left(\frac{m'}{s'}\right) \\ h\left(\frac{a}{s}\frac{m'}{s'}\right) &= h\left(\frac{am'}{ss'}\right) = (ss')^{-1}\tilde{h}(am') = as^{-1}s'^{-1}\tilde{h}(m') = \frac{a}{s}h\left(\frac{m'}{s'}\right) \end{aligned}$$

for every $m, m' \in M$, $s, s' \in S$, $a \in A$. We have abused that $\tilde{h}$ is an A module homomorphism. Moreover, h also makes the diagram of A_S modules and A_S homomorphisms commute:

$$gh\left(\frac{m}{s}\right) = g(s^{-1}\tilde{h}(m)) = s^{-1}g\tilde{h}(m) = s^{-1}f(m) = f\left(\frac{m}{s}\right)$$

for every $m \in M$, $s \in S$. Here we have used that g and f are A_S homomorphisms and that $g\tilde{h} = f$ when restricted to M. Thus M_S is a projective A_S module, as desired.

References

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[2] S. Lang, *Algebra (Revised 3rd Edition)*, Springer-Verlag, 2002.

[3] M. F. Atiyah, I. G. Macdonald *Introducción al Álgebra Conmutativa*, Reverté, 1978 (2015 reprint).