Introduction to Commutative and Homological Algebra - Homework 5

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Exercise 1

Let A, B be commutative rings.

1. Show that $\mathrm{Spec}(A \times B) = \mathrm{Spec}(A) \coprod \mathrm{Spec}(B)$ as sets. For this, we proceed by double inclusion:

   $\supseteq$) Let $P \in \mathrm{Spec}(A)$, then $P \times B \in \mathrm{Spec}(A \times B)$ since being prime lies in the structure of P. Similarly if we let $Q \in \mathrm{Spec}(B)$, then $A \times Q \in \mathrm{Spec}(A \times B)$. Hence $\mathrm{Spec}(A) \coprod \mathrm{Spec}(B) \subseteq \mathrm{Spec}(A \times B)$ since the two types of prime ideals of $A \times B$ that we found are disjoint.

   $\subseteq$) Let $I \in \mathrm{Spec}(A \times B)$, say $I = I_1 \times I_2$. Then $A \times B / I_1 \times I_2$ is an integral domain, and:

   $$\frac{A \times B}{I_1 \times I_2} \cong \frac{A}{I_1} \times \frac{B}{I_2}$$

   so for the right hand side to be an integral domain we need either $I_1 = A$ or $I_2 = B$. If both of these fail, then there exist a non zero element $x \in A/I_1$ and a non zero element $y \in B/I_2$, meaning that $(x, 0), (0, y) \in A/I_1 \times B/I_2$ and $(x, 0)(0, y) = (0, 0)$, a contradiction. Hence we either have $I = I_1 \times B$ or $I = A \times I_2$. Since I is prime, we must have in the first case that I_1 and in the second case that I_2 is prime, so $I \in \mathrm{Spec}(A)$ in the first case and $I \in \mathrm{Spec}(B)$ in the second case. Since these two cases are clearly disjoint, we have $\mathrm{Spec}(A \times B) \subseteq \mathrm{Spec}(A) \coprod \mathrm{Spec}(B)$.

   Thus $\mathrm{Spec}(A \times B) = \mathrm{Spec}(A) \coprod \mathrm{Spec}(B)$ as desired.

2. Show that $\dim(A \times B) = \max\{\dim(A), \dim(B)\}$.

   Consider a strictly decreasing chain of prime ideals of $A \times B$, say $P_0 \supsetneq \cdots \supsetneq P_r$. By the discussed above, $P_0 = P_0^A \times B$ or $P_0 = A \times P_0^B$, so all the elements in the sequence are of the form $P_i^A \times B$, $i = 1, \ldots, r$ in the first case and of the form $A \times P_i^B$, $i = 1, \ldots, r$ in the second case. Hence they define $P_0^A \supsetneq \cdots \supsetneq P_r^A$ a strictly decreasing chain of prime ideals of A or $P_0^B \supsetneq \cdots \supsetneq P_r^B$ a strictly decreasing chain of prime ideals of B. Thus by definition $\dim(A \times B) \leq \max\{\dim(A), \dim(B)\}$.

   Similarly, given a strictly decreasing chain of prime ideals of A, say $P_0^A \supsetneq \cdots \supsetneq P_r^A$, by the discussed above they define $P_0^A \times B \supsetneq \cdots \supsetneq P_r^A \times B$ a strictly decreasing chain of prime ideals of $A \times B$. Thus by definition $\dim(A) \leq \dim(A \times B)$. A completely analogous reasoning for B yields $\dim(B) \leq \dim(A \times B)$, so $\max\{\dim(A), \dim(B)\} \leq \dim(A \times B)$.

   Thus $\dim(A \times B) = \max\{\dim(A), \dim(B)\}$ as desired.

Exercise 2

Let $f : A \longrightarrow B$ be a ring homomorphism and ${}^a f : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ the induced map. Show that if I is an ideal of A, then $({}^a f)^{-1}(V(I)) = V(f(I)B)$.

Notice that as sets we have:

$$\begin{aligned}
({}^a f)^{-1}(V(I)) &= \{Q \in \mathrm{Spec}(B) : {}^a f(Q) \in V(I)\} = \{Q \in \mathrm{Spec}(B) : f^{-1}(Q) \in V(I)\} \\
&= \{Q \in \mathrm{Spec}(B) : I \subseteq f^{-1}(Q)\} = \{Q \in \mathrm{Spec}(B) : f(I) \subseteq Q\} \\
&= \{Q \in \mathrm{Spec}(B) : f(I)B \subseteq Q\} = V(f(I)B)
\end{aligned}$$

where we have used that $f(I) \subseteq Q$ if and only if $I \subseteq f^{-1}(Q)$, that since Q is an ideal then $f(I) \subseteq Q$ implies $f(I)B \subseteq QB = Q$, and conversely since $1 \in B$ having $f(I)B \subseteq Q$ implies $f(I) \subseteq Q$.

Exercise 3

Let $f : A \longrightarrow B$ be a homomorphism of noetherian rings and ${}^a f : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ the induced map. Let M be a finitely generated B-module, considered to be an A-module via f. Show that ${}^a f(\mathrm{Ass}_B(M)) = \mathrm{Ass}_A(M)$. We first notice that since A and B are noetherian we have $\mathrm{Ass}_A(M) \neq \emptyset \neq \mathrm{Ass}_B(M)$, and then proceed by double inclusion.

$\subseteq$) Let $P \in \mathrm{Ass}_B(M)$, so it is prime and $P = \mathrm{ann}_B(x)$ for certain $x \in M$. Thus $f^{-1}(P)$ is prime, we see that $f^{-1}(P) = \mathrm{ann}_A(x)$ by double inclusion. Let $s \in f^{-1}(P)$, then there is $t \in P$ with $f(t) = s$, so $s \cdot x = f(t)x = sx = 0$, so $s \in \mathrm{ann}_A(x)$. Let $t \in \mathrm{ann}_A(x)$, this means that $0 = t \cdot x = f(t)x$ hence $f(t) \in \mathrm{ann}_B(x) = P$ so $t \in f^{-1}(P)$. This means that ${}^a f(P) = f^{-1}(P) = f^{-1}(\mathrm{ann}_B(x)) = \mathrm{ann}_A(x) \in \mathrm{Ass}_A(M)$ since it is prime.

$\supseteq$) There are a few possible approaches to this inclusion. Let $Q \in \mathrm{Ass}_A(M)$, so it is prime and $Q = \mathrm{ann}_A(x)$ for certain $x \in M$.

The first two naive candidates for ideals in B are $f(Q)$ and $\mathrm{ann}_B(x)$. However, the first one fails to be prime since f does not need to be surjective, and the second one fails since we cannot guarantee that it is prime (we know that it can be contained in a maximal element of $\{\mathrm{ann}_B(z) : z \in M,\ z \neq 0\}$ and that this maximal one is prime). If we try fixing this last argument by setting $\mathrm{ann}_B(x) \subseteq \mathrm{ann}_B(y)$ with $\mathrm{ann}_B(y)$ the maximal ideal mentioned above, hence prime, then by a similar argument as in the inclusion above we can see that $f^{-1}(\mathrm{ann}_B(y)) \supseteq \mathrm{ann}_A(x)$, but we cannot guarantee that $f^{-1}(\mathrm{ann}_B(y)) \subseteq \mathrm{ann}_A(x)$. We may also take a more categorical approach, allowing enough tools from Algebraic Geometry: the functor ${}^a : \mathrm{Ab} \longrightarrow \mathrm{AffSch}$ from the category of commutative rings to the category of affine schemes is an equivalence of categories[3, Tag 01HX]. Moreover, both are small categories having all its elements being sets. This immediately yields that ^{a} preserves the set theoretical properties, and it should mean that ${}^a f(\mathrm{Ass}_B(M)) \subset \mathrm{Ass}_A(M)$. Using the inverse functor, it would seem that the other inclusion would follow. However, these approaches do not use all the hypothesis and thus it is not a surprise that I could not make them work.

A more elaborate approach is to consider $S = A \setminus Q$, a multiplicative set since Q is prime, and since x is not zero (otherwise Q wouldn't be proper) we can consider $x/1 \in M_S$ which is not zero. Hence $\mathrm{ann}_{B_S}(x/1)$ is an ideal, and as we reasoned in the previous paragraph, it is contained in some associated prime $T \in \mathrm{Ass}_{B_S}(M_S)$. Since we know that prime ideals of B_S are in a one to one correspondence with prime ideals of B that do not contain S, we can write $T = P_S$ with P a prime ideal of B not containing S. Since B is noetherian and M is finitely generated, we know that $\mathrm{Ass}_{B_S}(M_S) = \mathrm{Ass}_B(M) \cap \mathrm{Spec}(B_S)$, and the proof of this equality of sets relies on the one to one correspondence $P_S \leftrightarrow P$, meaning that $P \in \mathrm{Ass}_B(M)$. Once we are here, writing $P = \mathrm{ann}_B(y)$, knowing that $\mathrm{ann}_{B_S}(x/1) \subseteq P_S$ and using the one to one correspondence we have ${}^a f(P) = f^{-1}(P) \supseteq Q$ by an analogous argument to one done above. However, I could not rigorously see why ${}^a f(P) = f^{-1}(P) \subseteq Q$; it seems reasonable because we are using all the hypothesis on Q to find P, and we would like to say that the one to one correspondence $P_S \leftrightarrow P$ tells us that $y = x$, but this doesn't seem rigorous.

Exercise 4

Let k be a field and $A = k[x, y]$.

1. Show that the ideal (x^2, y) of A is primary and conclude that $(x^2, xy) = (x) \cap (x^2, y)$ is a primary decomposition of (x^2, y).

   Note that $A/(x^2, y) \cong k \oplus kx$ as vector spaces, which only has x as zero divisors. Since $x^2 = 0$, x is nilpotent and hence (x^2, y) is primary. For a similar reason, $A/(x) \cong k[y]$ as vector spaces, and this has no zero divisors (and hence all its zero divisors are nilpotent since it is an empty condition), so (x) is primary.

   Moreover, the double inclusion in $(x^2, xy) = (x) \cap (x^2, y)$ is clear; $p \in (x^2, xy)$ means $p = x^2 p_{x^2} + xy p_{xy} \in (x) \cap (x^2, y)$ for some $p_{x^2}, p_{xy} \in k[x, y]$, while $p \in (x^2, y)$ means $p = x^2 p_{x^2} + y p_y$ for some $p_{x^2}, p_y \in k[x, y]$, so also having $p \in (x)$ means that we need p_y to have x as common factor in all its terms, so $p = x^2 p_{x^2} + yx\tilde{p}_y \in (x^2, xy)$ where $p_y = x\tilde{p}_y$ for some $\tilde{p}_y \in k[x, y]$.

   Hence $(x^2, xy) = (x) \cap (x^2, y)$ is a decomposition into primary ideals, so by definition a primary decomposition.

2. Find $\sqrt{(x^2, y)}$. Notice that $x^2 \in (x^2, y)$ so $x \in \sqrt{(x^2, y)}$ and $y \in (x^2, y)$ so $y \in \sqrt{(x^2, y)}$. Thus since $\sqrt{(x^2, y)}$ is an ideal, this guarantees that $A \setminus k = k[x, y] \setminus k \in \sqrt{(x^2, y)}$. Since every element in (x^2, y) has a variable in all of its terms, every power of it will have a variable in all of its terms and hence there are no constant terms in $\sqrt{(x^2, y)}$ so $k \cap \sqrt{(x^2, y)} = \emptyset$. This means that $\sqrt{(x^2, y)} = k[x, y] \setminus k = \left(\oplus_{i=1}^{\infty} kx^i\right) \oplus \left(\oplus_{i=1}^{\infty} ky^i\right)$ are the ring and the vector space structure respectively.

Exercise 5

Let k be a field and $A = k[x,y,z]/(xy - z^2)$. Let $I = (\overline{x}, \overline{z})$ and $J = I^2 = (\overline{x}^2, \overline{xz}, \overline{z}^2)$ ideals of A. Show that I is prime, $\sqrt{J} = I$ and J is not primary.

To see that I is prime, we consider:

$$\frac{A}{I} \cong \frac{A = k[x,y,z]/(xy - z^2)}{(\overline{x}, \overline{z})} \cong \frac{k[\overline{x}, \overline{y}, \overline{z}]}{(\overline{x}, \overline{z})} \cong k[\overline{y}] \cong k[y]$$

since quotient by $xy - z^2$ tells us that we can simplify all terms in z of power two or greater as terms of a single power of z and powers of xy, but it does not apply restrictions to x or y. Thus when we quotient further by $(\overline{x}, \overline{z})$ the relation transforms into $\overline{xy} - \overline{z}^2$ which is always satisfied since it is identically zero in A/I, meaning that $k[\overline{y}] \cong k[y]$ indeed. Since this is an integral domain, we have that I is prime.

To check that $\sqrt{J} = I$, we proceed by double inclusion. Let $p \in \sqrt{J}$, this means $p^n \in J = I^2 \subseteq I$ for some $n \in \mathbb{N}$. Since I is prime, we have that $p \in I$. Let $p \in I$, then $p^2 \in I^2 = J$ so $p \in \sqrt{J}$.

To see that J is not primary, consider the ring A/J. Notice that $\overline{x}$ and $\overline{y}$ are both non-zero since no relationship cancels them out. Moreover, we have that $\overline{xy} = \overline{z}^2 = 0$ in A/J so both $\overline{x}$ and $\overline{y}$ are zero divisors. We have that $\overline{x}^2 = 0$ so $\overline{x}$ is nilpotent. However, there is no relationship expressing anything about powers of $\overline{y}$, so we have that it is not a nilpotent element of A/J, and thus J is not primary.

Exercise 6

Let k be a field, $A = k[x, y, z]$, $P_1 = (x, y)$, $P_2 = (x, z)$, $P_3 = (x, y, z)$, $I = P_1P_2$. Show that $I = P_1 \cap P_2 \cap P_3^2$ is a primary decomposition of I.

First, we check the equality. We have $I = (x, y)(x, z) = (x^2, xz, yx, yz)$ and $P_3^2 = (x, y, z)(x, y, z) = (x^2, xy, xz, y^2, yz, z^2)$, meaning that:

$$\begin{aligned} P_1 \cap P_2 \cap P_3^2 &= (x, y) \cap (x, z) \cap (x^2, xy, xz, y^2, yz, z^2) \\ &= (x, y) \cap (x^2, xy, xz, yz, z^2) \cap (x, z) = (x^2, xy, xz, yz) \cap (x, y) \cap (x, z) \\ &= (x^2, xy, xz, yz) = I \end{aligned}$$

since intersecting (x^2, xy, xz, yz, z^2) with (x, z) removes the possibility of having y^2, and then intersecting with (x, y) removes the possibility of having z^2, and what remains is exactly I.

Second, we check that these are primary. Notice $A/P_1 \cong k[z]$ and $A/P_2 \cong k[y]$, which both are integral domains and thus do not have zero divisors, so P_1 and P_2 are primary. Moreover:

$$\frac{A}{P_3^2} \cong \frac{k[x, y, z]}{(x^2, xy, xz, y^2, yz, z^2)} \cong \frac{k \oplus kx \oplus ky \oplus kz}{(xy, xz, yz)}$$

where the last equality is as vector spaces, remembering the multiplicative structure that allows us to take quotient. The quotient by the relations xy, xz, yz means that $\overline{x}$, $\overline{y}$, $\overline{z}$ are zero divisors, and they are the "basic" building blocks of zero divisors, that is, any zero divisor in A/P_3^2 will be multiplications and sums of $\overline{x}$, $\overline{y}$, $\overline{z}$. These three are nilpotent since $x^2, y^2, z^2 \in P_3^2$ meaning that $\overline{x}^2 = \overline{y}^2 = \overline{z}^2 = 0$. Since multiplications and sums preserve being nilpotent (a nilpotent element multiplied by anything is still nilpotent and a finite sum of nilpotents is nilpotent), knowing that $\overline{x}$, $\overline{y}$, $\overline{z}$ are nilpotent is enough to obtain that all zero divisors in A/P_3^2 are nilpotent. Hence P_3^2 is primary.

References

[1] T. W. Hungerford, *Algebra*, Springer-Verlag, 2000.

[2] S. Lang, *Algebra (Revised 3rd Edition)*, Springer-Verlag, 2002.

[3] The Stacks Project `stacks.math.columbia.edu`.