Representations of Finite Groups - Homework 1

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Exercise 1.1.4.

Show that each multiplicative character of G contains the commutator subgroup $[G, G]$ in its kernel.

Since $[G, G]$ is generated by elements of the for $xyx^{-1}y^{-1}$ for $x, y \in G$, given $\rho : G \longrightarrow K^*$ a multiplicative character of G, it is enough to check that $\rho(xyx^{-1}y^{-1}) = 1$ for every $x, y \in G$. This is clear because:

$$\rho(xyx^{-1}y^{-1}) = \rho(x)\rho(y)\rho(x^{-1})\rho(y^{-1}) = \rho(x)\rho(x^{-1})\rho(y)\rho(y^{-1}) = \rho(1) = 1$$

since K being a field ensures commutativity and ρ being a homomorphism ensures the rest of equalities. This proves the desired result.

Exercise 1.1.9.

Show that $\mathrm{M}_n(\mathbb{K})$ has no two sided ideals except for the two obvious ones, namely $\{0\}$ and $\mathrm{M}_n(\mathbb{K})$.

It is a standard algebra fact (see [1, Chapter III, Section 2, Exercise 8]) that for any commutative ring with identity R the two sided ideals of $\mathrm{M}_n(R)$ are of the form $\mathrm{M}_n(I)$ with I being and ideal of R. Applying this, since $\mathbb{K}$ is a field it only has ideals $\{0\}$ and $\mathbb{K}$ it means that $\mathrm{M}_n(\mathbb{K})$ has only as two sided ideals $\{0\}$ and $\mathrm{M}_n(\mathbb{K})$ respectively, as we desired.

Exercise 1.1.15.

Identify 1_g with the function whose value at g is 1 and which vanishes everywhere else. Under this identification, show that the two definitions of the group algebra given are equivalent: the first is that $K[G]$ has as K vector space the basis $\{1_g : g \in G\}$ and multiplication $1_g 1_h = 1_{gh}$ (we will denote this as $K[G]_1$), the second is that $K[G] = \{f : G \longrightarrow K\}$ as K vector space with multiplication the convolution $f_1 * f_2(g) = \sum_{x,y \in G | xy=g} f_1(x) f_2(y)$ (we will denote this as $K[G]_2$).

We define:

$$\begin{array}{rccc} \varphi & : & K[G]_1 & \longrightarrow & K[G]_2 \\ & & 1_g & \longrightarrow & f_g \end{array} \quad \text{with} \quad f_g(x) = \begin{cases} 1 \text{ if } x = g, \\ 0 \text{ if } x \neq g \end{cases}$$

and then extend by linearity, which is well defined since we are in a K vector space. Now φ is a K algebra homomorphism since by definition $\varphi(1_g + 1_h) = \varphi(1_g) + \varphi(1_h)$ for all $g, h \in G$ and $\varphi(k1_g) = k\varphi(1_g)$ for all $k \in K$ and $g \in G$, and moreover for every $g, h \in G$ we have:

$$\varphi(1_g 1_h) = \varphi(1_{gh}) = f_{gh} = f_g * f_h = \varphi(1_g) * \varphi(1_h)$$

where the only equality that needs additional justification is the third, which holds because for every $x \in G$:

$$f_g * f_h(x) = \sum_{\substack{y,z \in G \\ yz=x}} f_g(y) f_h(z) = \begin{cases} 1 \text{ if } y = g \text{ and } z = h, \\ 0 \text{ if } y \neq g \text{ or } z \neq h, \end{cases} = \begin{cases} 1 \text{ if } x = gh, \\ 0 \text{ if } x \neq gh \end{cases} = f_{gh}(x).$$

We define:

$$\begin{array}{rcccc} \phi & : & K[G]_2 & \longrightarrow & K[G]_1 \\ & & f_g & \longrightarrow & \sum_{g \in G} f(g) 1_g \end{array}$$

and then extend by linearity. This extension is well defined since we are in a K vector space, and ϕ is indeed defined over $K[G]_2$ since given $x \in G$ we have $f(x) = \sum_{g \in G} f(g) f_g(x)$ a decomposition of f in terms of f_g for $g \in G$. Now ϕ is a K algebra homomorphism since by definition $\phi(f_g + f_h) = \phi(f_g) + \phi(f_h)$ for all $g, h \in G$ and $\phi(k f_g) = k\phi(f_g)$ for all $k \in K$ and $g \in G$, and moreover for every $g, h \in G$ we have:

$$\begin{aligned} \phi(f_g * f_h) &= \sum_{l \in G} f_g * f_h(l) 1_l = \sum_{l \in G} \sum_{\substack{x,y \in G \\ xy=l}} f_g(x) f_h(y) 1_l \\ &= f_g(g) f_h(h) 1_{gh} = 1_{gh} = 1_g 1_h = \phi(f_g)\phi(f_h) \end{aligned}$$

where the only equality that may need additional justification is the third, which results noticing that by definition of f_g and f_h we have that $f_g(x) f_h(y)$ is always zero except when $x = g$ and $y = h$.

Notice that φ and ϕ are inverses from each other:

$$\varphi(\phi(f_g)) = \varphi(1_g) = f_g, \quad \phi(\varphi(1_g)) = \phi(f_g) = 1_g,$$

and thus they are bijective and hence the definitions are isomorphic, hence equivalent, as desired.

Exercise 1.1.16.

Let $n > 1$ be an integer. Show that $K[\mathbb{Z}_n]$ is isomorphic to $K[t]/(t^n - 1)$ as an algebra.

We define:

$$\begin{array}{cccc} \phi & : & K[t]/(t^n-1) & \longrightarrow & K[\mathbb{Z}_n] \\ & & t^m & \longrightarrow & 1_m \end{array}$$

and then extend by linearity. This extension is well defined since we are in a K vector space. Thus notice that by definition we have $\phi(t^m + t^l) = \phi(t^m) + \phi(t^l)$ for all $m, l \in \mathbb{N}$ and $\phi(kt^m) = k\varphi(t^m)$ for all $k \in K$ and $m \in \mathbb{Z}$, meaning that $\phi(t^n - 1) = \phi(t^n) - \phi(1) = 1_n - 1_0 = 1_0 - 1_0 = 0$ since $n \equiv 0$ in $\mathbb{Z}_n$. Thus ϕ is indeed defined over $K[t]/(t^n - 1)$ since given $p, q \in K[t]$ such that $p = q + r(t^n - 1)$ for some $r \in K$ we have $\phi(p) = \phi(q + r(t^n - 1)) = \phi(q) + r\phi(t^n - 1) = \phi(q)$. Moreover ϕ is a K algebra homomorphism since for every $m, l \in \mathbb{N}$ we have:

$$\phi(t^m t^l) = \phi(t^{m+l}) = 1_{m+l} = 1_m 1_l = \phi(t^m)\phi(t^l)$$

where the notation of $\mathbb{Z}_n$ being additive explains the sums instead of products.

Considering the above, notice that $K[t]/(t^n-1)$ has as basis the ordered set $\{1, t, \ldots, t^{n-1}\}$ and $K[\mathbb{Z}_n]$ has as basis the ordered set $\{1_0, 1_1, \ldots, 1_{n-1}\}$. Moreover, ϕ establishes a one to one correspondence between them (preserving the order), hence it is injective and surjective, and thus it is a K algebra isomorphism. This proves the desired result.

Exercise 1.1.18.

If $K[G]$ is viewed as the space of K valued functions on G, we prove that $(L(h)f)(x) = f(h^{-1}x)$ and $(R(h)f)(x) = f(xh)$ for all $f \in K[G]$ and $h, x \in G$.

Notice:

$$\begin{aligned}
(L(h)f)(x) &= L(h)\left(\sum_{g\in G} f(g)f_g\right)(x) = \left(\sum_{g\in G} f(g)f_{hg}\right)(x) \\
&= \left(\sum_{g\in G} f(h^{-1}g)f_g\right)(x) = f(h^{-1}x), \\
(R(h)f)(x) &= R(h)\left(\sum_{g\in G} f(g)f_g\right)(x) = \left(\sum_{g\in G} f(g)f_{gh^{-1}}\right)(x) \\
&= \left(\sum_{g\in G} f(gh)f_g\right)(x) = f(xh),
\end{aligned}$$

proving what we desired.

Exercise 1.2.4.

Let $G = \mathbb{Z}_2$ and K a field of characteristic 2. Show that the subspace of $K[G]$ spanned by $1_0 + 1_1$ is the only non trivial proper invariant subspace for the left regular representation of G.

As we proved in class, $K[G]_0$ has an invariant complement in $K[G]$ with the left regular representation if and only if the characteristic of the field does not divide the order of the group. Since 2 is both the order of G and the characteristic of K, we have that $K[G]_0$ does not have an invariant complement and it is 1 dimensional. Since $K[G]$ has dimension 2, any other 1 dimensional invariant subspace of $K[G]$ not intersecting $K[G]_0$ will be an invariant complement of the latter, and thus there are no such spaces. Moreover $w = 1_0 + 1_1 \in K[G]_0$ since $\sum_{g \in G} w(g) = w(0) + w(1) = 1 + 1 = 0$ since K has characteristic 2, and since $\langle w \rangle_K$ is 1 dimensional we must have that $\langle w \rangle_K = K[G]_0$ and thus by the above it is the only proper invariant subspace that is not trivial.

Exercise 1.2.5.

Show that if every representation of a group is a sum of 1 dimensional invariant subspaces, then the group is abelian.

We will use the left regular representation and the fact that each multiplicative character of G contains $[G,G]$ in its kernel. If $K[G]$ is a sum of 1 dimensional invariant subspaces, this means that $K[G] \equiv V_1 \oplus \overset{n}{\cdots} \oplus V_n \equiv K \oplus \overset{n}{\cdots} \oplus K$ as K vector spaces ($V_i \equiv K$ as K vector spaces for $i = 1, \dots, n$), and thus $L : G \longrightarrow \mathrm{GL}(K[G])$ can be decomposed into $L|_{V_i} : G \longrightarrow \mathrm{GL}(V_i) \cong K^*$ because V_i is 1 dimensional for $i = 1, \dots, n$. Each of this restrictions is a multiplicative character, and thus its kernel contains $[G,G]$. In particular by this decomposition $[G,G]$ is contained in the kernel of L. Thus fixing $g, h \in G$ we have for all $x \in G$:

$$1_{g^{-1}h^{-1}ghx} = L(g^{-1}h^{-1}gh)(1_x) = \mathrm{id}_{K[G]}(1_x) = 1_x$$

meaning that multiplying by 1_{hg} on both sides we obtain $1_{gh}1_x = 1_{hg}1_x$ for all $x \in G$, so in particular using the neutral element of G we have that $1_{gh} = 1_{hg}$. Since this is an equality as indicators, they both take value 1 in the same value, and since the left is gh and the right is hg, we obtain that $gh = hg$ for any two $g, h \in G$, meaning that G is commutative, as desired.

Exercise 1.2.8.

Every simple module of a finite dimensional K algebra is finite dimensional.

Let (ρ, V) be a simple module of R a finite dimensional K algebra with $\rho : R \longrightarrow \mathrm{End}_K(V)$. Since R is finite dimensional, we can find $\{v_i\}_{i=1}^n$ a K basis as vector space. We wish to find a generating set of V using this basis: our only option is to use ρ and apply the resulting endomorphisms to a non zero vector of V. Let $v \in V$ be a non zero vector, which exists can because V is simple (and thus not zero), consider $\langle\{\rho(v_i)v\}_{i=1}^n\rangle_K$. Notice that this is invariant since for any $i, j = 1, \ldots, n$ we have:

$$\rho(v_j)(\rho(v_i)v) = \rho(v_j v_i)v = \sum_{l=1}^{n} k_l \rho(v_l)v \in \langle\{\rho(v_i)v\}_{i=1}^n\rangle_K$$

where we have used that since $\{v_i\}_{i=1}^n$ is a K basis of R, we may write $v_j v_i = \sum_{l=1}^n k_l v_l$ with $k_l \in K$ for all $l = 1, \ldots, n$, and indeed it is enough to check only the basis of R since the rest follows by linearity. Thus $\langle\{\rho(v_i)v\}_{i=1}^n\rangle_K$ is an invariant subspace and it is not zero since $\rho(1_R)v = v$ which is not zero. Since V is simple, we must have that $\langle\{\rho(v_i)v\}_{i=1}^n\rangle_K = V$ and thus we found a generating set for V of size n, meaning that V has dimension n or less, in particular it is finite dimensional.

Exercise 1.2.12.

The kernel of an intertwiner is an invariant subspace of its domain, and the image is an invariant subspace of its codomain.

Denote (ρ_1, V_1), (ρ_2, V_2) and $T : V_1 \longrightarrow V_2$ the representations of a group G and the intertwiner. We have $\mathrm{Ker}(T) \subset V_1$, and given $v \in \mathrm{Ker}(T)$ then for every $g \in G$ we have $T(\rho_1(g)v) = \rho_2(g)Tv = \rho_2(0) = 0$ and thus $\rho_1(g)v \in \mathrm{Ker}(T)$, so it is indeed an invariant subspace. We have $\mathrm{Im}(T) \subset V_2$, and given $v \in \mathrm{Im}(T)$ then there is $w \in V_1$ with $T(w) = v$ and then for every $g \in G$ we have $\rho_2(g)v = \rho_2(g)Tw = T(\rho_1(g)w)$ with $\rho_1(g)w \in V_1$ and thus $\rho_2(g)v \in \mathrm{Im}(T)$, so it is indeed an invariant subspace.

References

[1] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.