Representations of Finite Groups - Homework 2

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February 7th, 2018

Exercise 1.3.10.

Let R a $\mathbb{K}$ algebra for an algebraically closed field. Show that a completely reducible R module V is simple if and only if $\dim \operatorname{End}_R(V) = 1$. Since V is completely reducible, we may write $V \cong V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r}$ for $m_i \in \mathbb{N}$, V_i simple R modules for all $i = 1, \ldots, r$, pairwise not isomorphic.

$\Rightarrow$) If V is simple, we have $r = 1$ and $m = 1$ (otherwise we have a contradiction with the definition of simple), thus by [1, Theorem 1.3.8. (p. 10)] we have $\dim \operatorname{End}_R(V) = 1$.

$\Leftarrow$) If $1 = \dim \operatorname{End}_R(V) = \sum_{i=1}^r m_i^2$, a sum of positive (or zero) integers, we must have that all but one summand is zero, thus $r = 1$ and since $m_1 \in \mathbb{N}$ we have $m_1 = 1$. Hence $V \cong V_1$, which is simple.

Exercise 1.3.11.

Let $\mathbb{K}$ algebraically closed, V simple and W completely reducible. Prove that $\dim \operatorname{Hom}_R(V, W)$ is the multiplicity of V in W.

By hypothesis both V and W are completely reducible, so in the notation above we may say that their decomposition into simple modules is $V \cong V_1$ and $W \cong V_1^{\oplus n_1} \oplus \cdots \oplus V_r^{\oplus n_r}$ with $n_i \in \mathbb{N}$ for all $i = 1, \ldots, r$. Then by [1, Theorem 1.3.5. (p. 9)] we have $\dim \operatorname{Hom}_R(V, W) = n_1$ which is the multiplicity of V in W.

Exercise 1.3.12.

Let $\mathbb{K}$ algebraically closed, prove that a completely reducible R module V has a multiplicity free decomposition if and only if $\mathrm{End}_R(V)$ is commutative. We will use the notation above, say $V \cong V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r}$ for $m_i \in \mathbb{N}$, V_i simple R modules for all $i = 1, \ldots, r$, pairwise not isomorphic.

$\Rightarrow$) If V has a multiplicity free decomposition, then $m_i = 1$ for all $i = 1, \ldots, r$. By [1, Theorem 1.3.6. (p. 9)] we have $\mathrm{End}_R(V) \cong \oplus_{i=1}^r M_1(\mathbb{K}) \cong \oplus_{i=1}^r \mathbb{K}$. Since the algebra multiplication is componentwise and $\mathbb{K}$ is a field, this is a commutative algebra.

$\Leftarrow$) In the general form for V, by [1, Theorem 1.3.6. (p. 9)] we have $\mathrm{End}_R(V) \cong \oplus_{i=1}^r M_{m_i}(\mathbb{K})$. Since this algebra is commutative by hypothesis, we have $m_i = 1$ for all $i = 1, \ldots, r$; if $m_i \geq 2$ for some $i = 1, \ldots, r$, then $M_{m_i}(\mathbb{K})$ is a matrix algebra, which is not commutative:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which is in $M_2(\mathbb{K})$, and appending 0 to fill up the remaining dimensions (if any) we obtain two non commuting elements in $M_{m_i}(\mathbb{K})$ and thus in $\mathrm{End}_R(V)$, a contradiction. Hence V has a multiplicity free decomposition.

Exercise 1.3.13.

Let $\mathbb{K}$ algebraically closed and V, W completely reducible finite dimensional R modules with $\dim \operatorname{End}_R(V) = \dim \operatorname{Hom}_R(V,W) = \dim \operatorname{End}_R(W)$. Prove that $V \cong W$.

In the notation above, write their decomposition into simple modules as $V \cong\cong V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r}$ and $W \cong V_1^{\oplus n_1} \oplus \cdots \oplus V_r^{\oplus n_r}$ with $m_i, n_i \in \mathbb{N}$ for all $i = 1, \ldots, r$. Then by [1, Theorem 1.3.5. (p. 9)] and [1, Theorem 1.3.8. (p. 10)] we have:

$$\begin{aligned}
\begin{cases} \dim \operatorname{End}_R(V) = \dim \operatorname{Hom}_R(V,W) \\ \dim \operatorname{Hom}_R(V,W) = \dim \operatorname{End}_R(W) \end{cases} &\implies \begin{cases} \sum_{i=1}^r m_i^2 = \sum_{i=1}^r m_i n_i \\ \sum_{i=1}^r m_i n_i = \sum_{i=1}^r n_i^2 \end{cases} \\
&\implies \begin{cases} \sum_{i=1}^r m_i(m_i - n_i) = 0 \\ \sum_{i=1}^r n_i(m_i - n_i) = 0 \end{cases} \\
&\implies \sum_{i=1}^r (m_i - n_i)^2 = \sum_{i=1}^r (m_i - n_i)(m_i - n_i) = 0
\end{aligned}$$

and since $(m_i - n_i)^2 \in \mathbb{N}$, we must have $m_i - n_i = 0$ for the sum to add up to 0, and thus $m_i = n_i$ so that $V \cong V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r} = V_1^{\oplus n_1} \oplus \cdots \oplus V_r^{\oplus n_r} = W$.

Exercise 1.3.14.

Let $V \cong\cong V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r}$ and $W \cong V_1^{\oplus n_1} \oplus \cdots \oplus V_r^{\oplus n_r}$ with $m_i, n_i \in \mathbb{N}$ and V_i simple R module for all $i = 1, \ldots, r$ pairwise non isomorphic. Prove that $V \cong W$ if and only if $m_i = n_i$ for all $i = 1, \ldots, r$.

$\Rightarrow$) If $V \cong W$, then using Exercise 1.3.11. we see that:

$$m_i = \dim \operatorname{Hom}_R(V_i, V) = \dim \operatorname{Hom}_R(V_i, W) = n_i$$

for all $i = 1, \ldots, r$ as desired.

$\Leftarrow$) By [1, Theorem 1.3.5. (p. 9)] and [1, Theorem 1.3.8. (p. 10)] we have:

$$\dim \operatorname{End}_R(V) = \sum_{i=1}^{r} m_i^2, \quad \dim \operatorname{Hom}_R(V, W) = \sum_{i=1}^{r} m_i n_i, \quad \dim \operatorname{End}_R(W) = \sum_{i=1}^{r} n_i^2$$

so if $m_i = n_i$ for all $i = 1, \ldots, r$, these are all equal. Thus by Exercise 1.3.13. we have $V \cong W$.

Exercise 1.3.15.

Let $\mathbb{K}$ algebraically closed, V_i pairwise non isomorphic simple R modules for $i = 1, \dots, r$. Show that every invariant subspace of $V = V_1 \oplus \cdots \oplus V_r$ is of the form $V_{i_1} \oplus \cdots \oplus V_{i_k}$ for some $1 \leq i_1 < \cdots < i_k \leq r$. Give an example where $V^{\oplus n}$ has infinitely many invariant subspaces when $\mathbb{K}$ is infinite.

Let W be an invariant subspace of V. Since V_i is simple, we either have $V_i \subset W$ or $V_i \cap W = \{0\}$ for every $i = 1 \dots, r$. If $V_i \subset W$, since it has multiplicity 1 in V by Exercise 1.3.11. we have $\dim \operatorname{Hom}_R(V_i, V) = 1$ and thus $\operatorname{Hom}_R(V_i, V) \cong \mathbb{K}\iota_i$ as vector space (and algebra with convolution as multiplication) for some non-zero intertwiner $\iota_i : V_i \longrightarrow V$. In particular just having V_i in W does not alter being invariant, without the need of adding anything else, and $\operatorname{Im}(\iota_i) \cong V_i$ since it is simple. Thus let $1 \leq i_1 < \cdots < i_k \leq r$ be such that $V_{i_j} \subset W$ for $j = 1, \dots, k$, then since $V_{i_j} \cap V_{i_l} = \{0\}$ for $j \neq l$ because they are pairwise not isomorphic, and $W \subset V_1 \oplus \cdots \oplus V_r$ so $W = V_{i_1} + \cdots + V_{i_k}$, we must have $W \cong V_{i_1} \oplus \cdots \oplus V_{i_k}$ as desired.

Consider now $\mathbb{K} = \mathbb{C}$ and $R = \mathbb{C}[x]/(x^2 - 1)$ acting on $\mathbb{C}^2$ via:

$$\begin{array}{cccc} \rho & : & R & \longrightarrow & \operatorname{End}_R \mathbb{C}^2 \\ & & x & \longmapsto & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

and extending by linearity. This obviously defines an R module structure on $\mathbb{C}^2$, and by appending the identity to fill up as many dimensions as we need it is readily generalized to define an R module structure on $\mathbb{C}^n$ for $n \geq 2$; we will work with $n = 2$ and by this the result will follow. Now given any $v \in \mathbb{C}^2$, it defines $\langle v \rangle_{\mathbb{C}} \subset \mathbb{C}^2$ an invariant subspace with respect to this action, since the action just changes its sign. Hence $\mathbb{C}^2$ has as many invariant subspaces as vectors, and since $\mathbb{C}$ is infinite the number of vectors and invariant subspaces is infinite.

Exercise 1.4.4.

Let A be a $m \times m$ matrix with entries in $\mathbb{K}$ an algebraically closed field. Suppose that $A^n = 1$ with n not divisible by the characteristic of $\mathbb{K}$. Show that A is diagonalizable.

Consider $G = \{A^k : k \in \mathbb{N}\}$ the finite abelian group generated by iterations of A. Now:

$$\begin{array}{rcccc} \rho & : & G & \longrightarrow & \mathrm{End}_{\mathbb{K}}(\mathbb{K}^m) \\ & & A & \longrightarrow & A \end{array}$$

is the canonical inclusion and thus clearly a representation of G. Since $|G| = n$ is not divisible by the characteristic of $\mathbb{K}$, by [1, Theorem 1.4.3. (p. 11)] we have that $\mathbb{K}^m$ is completely reducible. Moreover, by Exercise 1.2.9. we have that every simple representation of G is of dimension 1. This means that $\mathbb{K}^m \cong V_1 \oplus \cdots \oplus V_m$ with V_i simple of dimension 1 for all $i = 1, \ldots, m$. Thus taking one $v_i \in V_i$ for each $i = 1, \ldots, r$ we obtain a basis of $\mathbb{K}^m$, and since these are simple, with respect to this base A is diagonal, as desired.

Exercise 1.5.5.

Prove that the map $\phi : M_n(\mathbb{K}) \longrightarrow M_n(\mathbb{K})^{\text{op}}$ given by $\phi(A) = A^T$ for every $A \in M_n(\mathbb{K})$ is an isomorphism of $\mathbb{K}$ algebras.

We will use the standard Linear Algebra facts that for any $A, B \in M_n(\mathbb{K})$ and $\alpha \in \mathbb{K}$ it holds $(A+B)^T = A^T + B^T$, $(\alpha A)^T = \alpha A^T$, $(AB)^T = B^T A^T$ and $(A^T)^T = A$. Now clearly:

$$\begin{aligned}
\phi(\alpha A) &= (\alpha A)^T = \alpha A^T = \alpha \phi(A) \\
\phi(A+B) &= (A+B)^T = A^T + B^T = \phi(A) + \phi(B) \\
\phi(AB) &= (AB)^T = B^T A^T = \phi(B)\phi(A) = \phi(A) \cdot_{\text{op}} \phi(B)
\end{aligned}$$

so it is a $\mathbb{K}$ algebra homomorphism. If there is $A \in M_n(\mathbb{K})$ with $\phi(A) = 0$ this means $A^T = 0$ and thus all the entries in A^T are zero, but these are the same entries that in A, thus $A = 0$, so ϕ is injective. If we are given $A \in M_n(\mathbb{K})^{\text{op}}$, we have $A \in M_n(\mathbb{K})$ since they have the same vector space structure, thus $A^T \in M_n(\mathbb{K})$ and $\phi(A^T) = (A^T)^T = A$, so ϕ is surjective. Thus ϕ is bijective and indeed an isomorphism of $\mathbb{K}$ algebras.

Exercise 1.5.6.

Let G be a group, prove that the map $\phi : \mathbb{K}[G] \longrightarrow \mathbb{K}[G]^{\mathrm{op}}$ given by $\phi(1_g) = 1_{g^{-1}}$ for every $g \in G$ is an isomorphism of $\mathbb{K}$ algebras.

We first note that as it is, this is not a well defined map since sums and product by scalars are not defined, hence we assume that the definition is meant to be extended by linearity, which is possible since $\mathbb{K}[G]$ has a vector space structure and we defined ϕ on a basis. Hence we automatically have linearity; given $g, h \in G$ and $\alpha \in \mathbb{K}$ then $\phi(\alpha 1_g) = \alpha\phi(1_g)$ and $\phi(1_g + 1_h) = \phi(1_g) + \phi(1_h)$. Moreover:

$$\phi(1_g 1_h) = \phi(1_{gh}) = 1_{(gh)^{-1}} = 1_{h^{-1}g^{-1}} = 1_{h^{-1}}1_{g^{-1}} = \phi(1_h)\phi(1_g) = \phi(1_g) \cdot_{\mathrm{op}} \phi(1_h)$$

so it indeed is a $\mathbb{K}$ algebra homomorphism. We define what will be the inverse as the map $\psi : \mathbb{K}[G]^{\mathrm{op}} \longrightarrow \mathbb{K}[G]$ given by $\phi(1_g) = 1_{g^{-1}}$ and extending by linearity, possible since $\mathbb{K}[G]^{\mathrm{op}}$ has a vector space structure. Again, linearity comes by definition; given $g, h \in G$ and $\alpha \in \mathbb{K}$ then $\psi(\alpha 1_g) = \alpha\psi(1_g)$ and $\psi(1_g + 1_h) = \psi(1_g) + \psi(1_h)$. Moreover:

$$\psi(1_g \cdot_{\mathrm{op}} 1_h) = \psi(1_h 1_g) = \psi(1_{hg}) = 1_{(hg)^{-1}} = 1_{g^{-1}h^{-1}} = 1_{g^{-1}}1_{h^{-1}} = \psi(1_g)\psi(1_h)$$

so it indeed is a $\mathbb{K}$ algebra homomorphism. They clearly are inverses from each other since for any $g \in G$:

$$\psi \circ \phi(1_g) = \psi(1_{g^{-1}}) = 1_{(g^{-1})^{-1}} = 1_g$$
$$\phi \circ \psi(1_g) = \psi(1_{g^{-1}}) = 1_{(g^{-1})^{-1}} = 1_g$$

and thus ϕ is bijective and an isomorphism of $\mathbb{K}$ algebras.

References

[1] A. Prasad, *Representation Theory: A Combinatorial Viewpoint*, Cambridge studies in advanced mathematics, 2015.

[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.