Representations of Finite Groups - Homework 3

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Exercise 1.5.9.

For each $V \subseteq \mathbb{K}^n$ linear subspace, define M_V the set of all matrices whose rows (as elements of $\mathbb{K}^n$) lie in V.

1. Prove that M_V is an invariant subspace of the left regular $M_n(\mathbb{K})$ module of dimension $n\dim_{\mathbb{K}}(V)$.

   We can consider elements $B \in M_V$ as $B = [b_1, \cdots, b_n]^T$. We clearly have that this is a subspace of $M_n(\mathbb{K})$ since $0_{n\times n} \in M_n(\mathbb{K})$ because $0 \in V$, if $C \in M_V$ as $C = [c_1, \cdots, c_n]^T$ then $B + C = [b_1 + c_1, \dots, b_n + c_n] \in M_V$ and if $\alpha \in \mathbb{K}$ then $\alpha B = [\alpha b_1, \dots, \alpha b_n] \in M_V$, all three true because V is a linear subspace.

   To see that it is invariant, given any $A \in M_n(\mathbb{K})$ with entries a_{ij} for $i, j = 1, \dots, n$, notice how:
$$L(A)(B) = AB = \begin{bmatrix} \sum_{i=1}^n a_{1i}b_i \\ \vdots \\ \sum_{i=1}^n a_{ni}b_i \end{bmatrix}$$
   and since V is a linear subspace of $\mathbb{K}$, we have that $\sum_{i=1}^n a_{ji}b_i \in V$ for all $j = 1, \dots, n$ and thus $L(A)(B) \in M_V$ and M_V is invariant.

   To compute the dimension, let $V = \langle v_1, \dots, v_m\rangle$ be a basis, so $m \leq n$ is the dimension of V. Consider $E_i(v_j)$ the matrix with the vector v_j in the i-th row for $i = 1, \dots, n$ and $j = 1, \dots, n$, notice that there are $nm = n\dim_{\mathbb{K}}(V)$ of them. We now prove that they generate: let $B \in M_V$ as above, then $b_i = \sum_{j=1}^m \alpha_{ij}v_j$ for some $\alpha_{ij} \in \mathbb{K}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$, so $B = \sum_{i=1}^n \sum_{j=1}^m \alpha_{ij}E_i(v_j)$ and hence these matrices indeed generate. We now prove that they are linearly independent: if we have $0_{n\times n} = \sum_{i=1}^n \sum_{j=1}^m \alpha_{ij}E_i(v_j)$ for some coefficients $\alpha_{ij} \in \mathbb{K}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. If we look at this equality row by row, this means that as elements of V we have: $0 = \sum_{j=1}^m \alpha_{ij}v_j$ for $i = 1, \dots, n$ and $j = 1, \dots, m$, and since this is a linear combination of the elements of the basis that adds up to zero, we must have that $\alpha_{ij} = 0$ for $i = 1, \dots, n$ and $j = 1, \dots, m$, thus the considered matrices are indeed linearly independent. This proves that $\{E_i(v_j)\}_{i=1,\dots,n}^{j=1,\dots,m}$ is a basis of M_V, thus $\dim_{\mathbb{K}}(M_V) = n\dim_{\mathbb{K}}(V)$ as desired.

2. Prove that every invariant subspace of the left regular $M_n(\mathbb{K})$ module is of the form M_V for some $V \subseteq \mathbb{K}^n$ linear subspace.

   Let $I \subseteq M_n(\mathbb{K})$ be an invariant subspace, we define:
$$V = \{b \in \mathbb{K}^n : \exists B \in I \text{ having } b \text{ as a row}\},$$
   we first notice that if an element $b \in V$, we can assume that it appears as the first row of a matrix B and the rest are 0: suppose $b = B_j$ the i-th element in $B = [b_1, \dots, b_n]^T$, consider E_{1i} the matrix with a 1 in the position $(1, i)$ and 0 elsewhere, then since I is an invariant subspace we have that $[b_i, 0 \dots, 0]^T = E_{1i}B = L(E_{1i})(B) \in I$, so we may take this matrix.

We clearly have that $M_V = I$ by definition, so we just need to prove that $V \subseteq \mathbb{K}^n$ is a linear subspace. For this, we clearly have that $0 \in V$ since we can multiply by $0_{n\times n} \in M_n(\mathbb{K})$, let $\alpha \in \mathbb{K}$ and $b \in V$ appearing in $B = [b, 0 \ldots, 0]^T$, then $[\alpha b, 0, \ldots, 0]^T = \alpha E_{11} B = L(\alpha E_{11})(B) \in I$ so $\alpha b \in V$, let $c \in V$ appearing in $C = [c, 0 \ldots, 0]^T$, then $[b + c, 0, \ldots, 0]^T = B + C \in I$ so $b + c \in V$. Then V is indeed a linear subspace.

3. Prove that M_V is simple if and only if V is one dimensional.

   $\Rightarrow$) If M_V is simple, suppose $\dim_{\mathbb{K}}(V) = 2$, say $V = \langle v_1, v_2 \rangle$ is a basis. Then $M_{\langle v_1 \rangle} \subseteq M_V$ is invariant by the first section above and proper since it is not zero since $0 \neq [v_1, 0 \ldots, 0]^T \in M_{\langle v_1 \rangle}$ and it is not everything since $[v_2, 0, \ldots, 0]^T \notin M_{\langle v_1 \rangle}$ since they are linearly independent. If $\dim_{\mathbb{K}}(V) \geq 2$, it always has a linear subspace of dimension 2 (take any two elements of the basis) and thus the above finds a proper invariant subspace, contradicting simplicity.

   $\Leftarrow$) If $\dim_{\mathbb{K}}(V) = 1$ we set $V = \langle v \rangle$, then every matrix $B \in M_V$ is of the form $B = [\alpha_1 v, \ldots, \alpha_n v]^T$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. Suppose $I \subset M_V$ is an invariant subspace, by the section above we can assume $I = M_W$ for some $W \subset \mathbb{K}^n$ linear subspace, that in fact $W \subseteq V$ by the definition of M_W and M_V, and that $B \in M_W$ is not zero. Now $[v, 0 \ldots, 0] = \alpha_1^{-1} E_{11} B = L(\alpha_1^{-1} E_{11})(B) \in M_W$ meaning that $v \in W$ and thus $V \subseteq W$.

4. Prove that M_V is isomorphic to M_W as an $M_n(\mathbb{K})$ module if and only if V and W have the same dimension.

   $\Rightarrow$) If $M_V \cong M_W$, we have $T : M_V \longrightarrow M_W$ an intertwiner that is an isomorphism, in particular a bijective linear map, so it preserves dimensions and thus $n \dim_{\mathbb{K}}(V) = \dim_{\mathbb{K}}(M_V) = \dim_{\mathbb{K}}(M_W) = n \dim_{\mathbb{K}}(W)$ so $\dim_{\mathbb{K}}(V) = \dim_{\mathbb{K}}(W)$.

   $\Leftarrow$) If $\dim_{\mathbb{K}}(V) = \dim_{\mathbb{K}}(W)$ we set $V = \langle v_1, \ldots, v_m \rangle$ and $W = \langle w_1, \ldots, w_m \rangle$ as basis, with $m \leq n$. Then we define:

$$\begin{array}{cccc} T & : & M_V & \longrightarrow & M_W \\ & & E_i(v_j) & \longmapsto & E_i(w_j) \end{array}$$

   and extend by linearity (they are vector spaces). This sends a basis of M_V to a basis of M_W in a bijective way, and is a linear transformation by construction. We now prove that it is an intertwiner, noticing that we only need to prove it on the elements of the basis, that is, check that for every $A \in M_n(\mathbb{K})$ we have that $AE_i(w_j) = L(A)T(E_i(v_j)) = T(L(A)(E_i(v_j)) = T(AE_i(v_j))$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$. For this:

$$\begin{aligned} T(AE_i(v_j)) &= T(A[0, \ldots, 0, \overset{i)}{v_j}, 0, \ldots, 0]^T) = T([a_{1i}v_j, \ldots, a_{ni}v_j]^T) \\ &= T(a_{1i}E_1(v_j) + \cdots + a_{ni}E_n(v_j)) = a_{1i}E_1(w_j) + \cdots + a_{ni}E_n(w_j) \\ &= [a_{1i}w_j, \ldots, a_{ni}w_j]^T = A[0, \ldots, 0, \overset{i)}{w_j}, 0, \ldots, 0]^T = AE_i(w_j). \end{aligned}$$

   Thus T is indeed an isomorphism of representations, as desired.

Exercise 1.6.2.

Let X, Y be two finite sets, construct an isomorphism $K[X] \otimes K[Y] \cong K[X \times Y]$.

We define:

$$\begin{array}{cccccccc} \overline{\phi} & : & K[X] \otimes K[Y] & \longrightarrow & K[X \times Y] & & & \\ & & (f, g) & \longrightarrow & \overline{\phi}(f, g) & : & X \times Y & \longrightarrow & \mathbb{K} \\ & & & & & & (x, y) & \longmapsto & f(x)g(x) \end{array}$$

and notice that it is bilinear since for any $f, \tilde{f} \in K[X]$, $g, \tilde{g} \in K[Y]$, $x \in X$, $y \in Y$ and $\alpha \in \mathbb{K}$ we have:

$$\begin{aligned} \overline{\phi}(f + \tilde{f}, g)(x, y) &= (f + \tilde{f})(x)g(y) = f(x)g(y) + \tilde{f}(x)g(y) = \overline{\phi}(f, g)(x, y) + \overline{\phi}(\tilde{f}, g)(x, y) \\ \overline{\phi}(f, g + \tilde{g})(x, y) &= f(x)(g(y) + \tilde{g}(y)) = f(x)g(y) + f(x)\tilde{g}(y) = \overline{\phi}(f, g)(x, y) + \overline{\phi}(f, \tilde{g})(x, y) \\ \overline{\phi}(\alpha f, g)(x, y) &= (\alpha f)(x)g(x) = \alpha f(x)g(x) \\ \overline{\phi}(f, \alpha g)(x, y) &= f(x)(\alpha g)(x) = f(x)\alpha g(x) = \alpha f(x)g(x) \\ \alpha\overline{\phi}(f, g)(x, y) &= \alpha f(x)g(y) \end{aligned}$$

and thus there exists a linear $\phi : K[X] \otimes K[Y] \longrightarrow K[X \times Y]$ by the Universal Property of the tensor product.

Notice that on the indicators $1_x \in K[X]$ for $x \in X$ and $1_y \in K[Y]$ for $y \in Y$ we have that $\overline{\phi}(1_x, 1_y) = 1_x 1_y = 1_{(x,y)}$ the indicator for $(x, y) \in X \times Y$. Thus $\phi(1_x \otimes 1_y) = 1_{(x,y)}$ for any $x \in X$ and $y \in Y$. This yields that ϕ is surjective since it is linear and has in the image all the indicators $1_{(x,y)} \in K[X \times Y]$ for $(x, y) \in X \times Y$. To check injectivity, notice that we may write $f \in K[X]$ as $f = \sum_{x \in X} f(x)1_x$ and $g \in K[Y]$ as $g = \sum_{y \in Y} f(y)1_y$, meaning that if $f \otimes g \in \ker(\phi)$ then:

$$0 = \phi(f \otimes g) = \left(\sum_{x \in X} f(x)1_x \right) \left(\sum_{y \in Y} f(y)1_y \right) = \sum_{(x,y) \in X \times Y} f(x)g(y)1_{(x,y)}$$

which means that $f(x)g(y) = 0$ for all $(x, y) \in X \times Y$ since $1_{(x,y)}$ ranging $(x, y) \in X \times Y$ form a basis of $K[X \times Y]$. This means that either $f(x) = 0$ for all $x \in X$ or $g(y) = 0$ for all $y \in Y$, as if we suppose that there exists $\tilde{x} \in X$ with $f(\tilde{x}) \neq 0$ then $f(\tilde{x})g(y) = 0$ for all $y \in Y$ implies $g(y) = 0$ for all $y \in Y$ since $\mathbb{K}$ is a field. Thus $f0$ or $g = 0$ respectively, and in either case $f \otimes g = 0$, obtaining injectivity.

Thus ϕ is a linear bijection, thus an isomorphism, as desired.

Exercise 1.6.3.

Show that if $S : V_1 \longrightarrow V_2$ and $T : W_1 \longrightarrow W_2$ are linear maps (and V_1, V_2, W_1, W_2 finite dimensional) then $\overline{\phi} : \mathrm{Hom}_{\mathbb{K}}(V_1, V_2) \times \mathrm{Hom}_{\mathbb{K}}(W_1, W_2) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes W_1, V_2 \otimes W_2)$ given by $\overline{\phi}(S, T) = S \otimes T$ induces an isomorphism $\phi : \mathrm{Hom}_{\mathbb{K}}(V_1, V_2) \otimes \mathrm{Hom}_{\mathbb{K}}(W_1, W_2) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes W_1, V_2 \otimes W_2)$.

Notice that $\overline{\phi}$ is bilinear since for any $S, \tilde{S} \in \mathrm{Hom}_{\mathbb{K}}(V_1, V_2)$, $T, \tilde{T} \in \mathrm{Hom}_{\mathbb{K}}(W_1, W_2)$ and $\alpha \in \mathbb{K}$ we have:

$$\begin{aligned}
\overline{\phi}(S + \tilde{S}, T) &= (S + \tilde{S}) \otimes T = S \otimes T + \tilde{S} \otimes T = \overline{\phi}(S, T) + \overline{\phi}(\tilde{S}, T) \\
\overline{\phi}(S, T + \tilde{T}) &= S \otimes (T + \tilde{T}) = S \otimes T + S \otimes \tilde{T} = \overline{\phi}(S, T) + \overline{\phi}(S, \tilde{T}) \\
\overline{\phi}(\alpha S, T) &= (\alpha S) \otimes T = \alpha(S \otimes T) \\
\overline{\phi}(S, \alpha T) &= S \otimes (\alpha T) = \alpha(S \otimes T) \\
\alpha\overline{\phi}(S, T) &= \alpha(S \otimes T)
\end{aligned}$$

and thus there exists a linear $\phi : \mathrm{Hom}_{\mathbb{K}}(V_1, V_2) \otimes \mathrm{Hom}_{\mathbb{K}}(W_1, W_2) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes W_1, V_2 \otimes W_2)$ by the Universal Property of the tensor product.

We first notice that the dimensions of the range and target of ϕ are the same:

$$\begin{aligned}
\dim_{\mathbb{K}}(\mathrm{Hom}_{\mathbb{K}}(V_1, V_2) \otimes \mathrm{Hom}_{\mathbb{K}}(W_1, W_2)) &= \dim_{\mathbb{K}}(\mathrm{Hom}_{\mathbb{K}}(V_1, V_2)) \dim_{\mathbb{K}}(\mathrm{Hom}_{\mathbb{K}}(W_1, W_2)) \\
&= \dim_{\mathbb{K}}(V_1) \dim_{\mathbb{K}}(V_2) \dim_{\mathbb{K}}(W_1) \dim_{\mathbb{K}}(W_2) \\
&= \dim_{\mathbb{K}}(V_1) \dim_{\mathbb{K}}(W_1) \dim_{\mathbb{K}}(V_2) \dim_{\mathbb{K}}(W_2) \\
&= \dim_{\mathbb{K}}(V_1 \otimes W_1) \dim_{\mathbb{K}}(V_2 \otimes W_2) \\
&= \dim_{\mathbb{K}}(\mathrm{Hom}_{\mathbb{K}}(V_1 \otimes W_1, V_2 \otimes W_2))
\end{aligned}$$

and thus to prove that it is bijective it suffices to check that it establishes a bijection between the elements of the basis of $\mathrm{Hom}_{\mathbb{K}}(V_1, V_2) \otimes \mathrm{Hom}_{\mathbb{K}}(W_1, W_2)$ and $\mathrm{Hom}_{\mathbb{K}}(V_1 \otimes W_1, V_2 \otimes W_2)$. For this, we set $V_1 = \langle v_1^1, \ldots, v_{n_1}^1 \rangle$, $V_2 = \langle v_1^2, \ldots, v_{n_2}^2 \rangle$, $W_1 = \langle w_1^1, \ldots, w_{m_1}^1 \rangle$, $W_2 = \langle w_1^2, \ldots, w_{m_2}^2 \rangle$ be the basis of the respective spaces. Then $\mathrm{Hom}_{\mathbb{K}}(V_1, V_2)$ has basis A_{ij} the matrix having 1 in the position (i, j) and 0 elsewhere, for $1 \leq i \leq n_2$ and $1 \leq j \leq n_1$, $\mathrm{Hom}_{\mathbb{K}}(W_1, W_2)$ has basis B_{kl} the matrix having 1 in the position (k, l) and 0 elsewhere, for $1 \leq k \leq m_2$ and $1 \leq l \leq m_1$. Thus $\mathrm{Hom}_{\mathbb{K}}(V_1, V_2) \otimes \mathrm{Hom}_{\mathbb{K}}(W_1, W_2)$ has the usual basis in terms of these two and $\mathrm{Hom}_{\mathbb{K}}(V_1 \otimes W_1, V_2 \otimes W_2)$ has basis C_{st} the matrix having 1 in the position (s, t) and 0 elsewhere, for $1 \leq s \leq n_2 m_2$ and $1 \leq t \leq n_1 m_1$, where in both cases the order of the basis elements is the usual one corresponding to the tensor product.

We will now fix $1 \leq i \leq n_2$, $1 \leq j \leq n_1$, $1 \leq k \leq m_2$, $1 \leq l \leq m_1$ and compute the map $\phi(A_{ij} \otimes B_{kl}) = A_{ij} \otimes B_{kl}$ applied to the elements of the basis of $V_1 \otimes W_1$: notice by the above that $A_{ij}(v_p^1) = \delta_{jp} v_i^2$ and $B_{kl}(w_q^1) = \delta_{lq} w_k^2$, so:

$$(A_{ij} \otimes B_{kl})(v_p^1 \otimes w_q^1) = A_{ij}(v_p^1) \otimes B_{kl}(w_q^1) = (\delta_{jp} v_i^2) \otimes (\delta_{lq} w_k^2)$$

and for this to be non zero we need $j = p$ and $l = q$, and in that case $A_{ij} \otimes B_{kl}$ is the matrix with a 1 in the position $(m_2(i-1) + k, m_1(j-1) + l)$. The way to notice this is that in matrix form, the row (that is the first component) is determined by the position of the basis element in the target space, while the column (that is the second component) is determined by the position of the basis element in the range space. Thus going to the element basis $v_i^1 \otimes w_k^1$, which is in the position $m_1(i-1)+k$, from the element basis $v_j^1 \otimes w_l^1$, which is in the position $m_1(j-1)+l$, requires the matrix with a 1 in the position $(m_2(i-1)+k, m_1(j-1)+l)$.

Thus the map ϕ is clearly a bijective one since we indeed have in its image all the elements of the basis of $\mathrm{Hom}_{\mathbb{K}}(V_1 \otimes W_1, V_2 \otimes W_2)$. Thus ϕ is a bijective linear transformation so indeed an isomorphism. Moreover, notice that the computation above establishes that $\mathrm{Im}(\phi)$ is indeed the usual definition of tensor product among matrices, otherwise known as Kronecker product. This will be extremely useful for the following exercise.

Exercise 1.6.4.

Show that if $S : V_1 \longrightarrow V_2$ and $T : W_1 \longrightarrow W_2$ are linear maps (and V_1, V_2, W_1, W_2 finite dimensional) then $\mathrm{Tr}(S \otimes T) = \mathrm{Tr}(S)\mathrm{Tr}(T)$.

Fixing the basis above (we must have $n_1 = n_2 = n$ and $m_1 = m_2 = m$ to compute the trace) and considering both S and T as matrices, say $S = (s_{ij})_{i,j=1,\ldots,n}$ and $T = (t_{kl})_{k,l=1,\ldots,m}$, then using that $S \otimes T$ is the Kronecker product, we find that:

$$\begin{aligned}\mathrm{Tr}(S \otimes T) = \mathrm{Tr}\begin{bmatrix} s_{11}T & \cdots & s_{1n}T \\ \vdots & & \vdots \\ s_{n1}T & \cdots & s_{nn}T \end{bmatrix} &= s_{11}\mathrm{Tr}(T) + \cdots + s_{nn}\mathrm{Tr}(T) \\ &= \left(\sum_{i=1}^{n} s_{ii}\right)\mathrm{Tr}(T) = \mathrm{Tr}(S)\mathrm{Tr}(T)\end{aligned}$$

as desired.

Exercise 1.6.8.

Let (ρ, V) and (σ, W) be representations of groups G and H respectively, then $(\rho' \boxtimes \sigma, V' \otimes W)$ is a representation of $G \times H$. Also, $\mathrm{Hom}_{\mathbb{K}}(V, W)$ is a representation of $G \times H$ via $\tau : G \times H \longrightarrow \mathrm{GL}(\mathrm{Hom}_{\mathbb{K}}(V, W))$ given by $\tau(g,h)(T) = \sigma(h) \circ T \circ \rho(g)^{-1}$. Show that the isomorphism $T : V' \otimes W \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V, W)$ induced by $\overline{T} : V' \times W \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V, W)$ given by $\overline{T}(\xi, y)(x) = \xi(x)y$ is an intertwiner of the representations of $G \times H$, and thus $V' \otimes W \cong \mathrm{Hom}_{\mathbb{K}}(V, W)$ as representations of $G \times H$.

For T to be an intertwiner we must have for every $(g,h) \in G \times H$ that $\tau(g,h) \circ T = T \circ (\rho' \boxtimes \sigma)(g,h)$, so given any $\xi \otimes y \in V' \otimes W$ and $x \in V$, it is enough to prove that $\tau(g,h) \circ T(\xi \otimes y)(x) = T \circ (\rho' \boxtimes \sigma)(g,h)(\xi \otimes y)(x)$. This is true:

$$\begin{aligned}
\tau(g,h) \circ T(\xi \otimes y)(x) &= (\sigma(h) \circ T(\xi \otimes y) \circ \rho(g)^{-1})(x) = \sigma(h)(T(\xi \circ y)(\rho(g)^{-1}(x))) \\
&= \sigma(h)(\xi(\rho(g)^{-1}(x))y) = \xi(\rho(g)^{-1}(x))\sigma(h)(y) \\
T \circ (\rho' \boxtimes \sigma)(g,h)(\xi \otimes y)(x) &= T((\rho'(g) \otimes \sigma(h))(\xi \otimes y))(x) = T(\rho'(g)(\xi) \otimes \sigma(h)(y))(x) \\
&= \rho'(g)(\xi)(x)\sigma(h)(y)
\end{aligned}$$

which are indeed equal since by definition ρ' is given by $\rho'(g)(\xi)(x) = \xi(\rho(g)^{-1}(x))$ for all $g \in G$, $\xi \in V'$ and $x \in V$. Thus we obtain the desired commutativity and thus T is an intertwiner.

References

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[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.