Representations of Finite Groups - Homework 4

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Exercise 1.7.4.

Let (ρ_i, V_i) for $i = 1, \dots, r$ be the representatives of the isomorphism classes of simple representations of a group G. Work out the isomorphism $\Phi : \oplus_{i=1}^r V_i \otimes V_i' \longrightarrow K[G]$ given by $\Phi(x_i \otimes \xi_i) = \sum_{g \in G} \xi(\rho(g)(x))$ for $G = \mathbb{Z}_3$ and $K = \mathbb{C}$. Then for $G = \mathbb{Z}_n$ and $K = \mathbb{C}$.

Notice that in the first case we can find three one dimensional representations $\rho_i : G \longrightarrow \mathbb{C}^\times \cong \mathrm{GL}(\mathbb{C})$ for $i = 0, 1, 2$ given by $\rho_i(1) = \alpha^i$ where $\alpha^3 = \rho_i(1)^3 = \rho_i(3) = \rho_i(0) = 1$, that is, α is a third root of unity (we will want α to be the primitive root of unity). Notice that one dimensional representations (ρ, V) are fully determined by the image of $1 \in \mathbb{Z}_n$ since $\rho(g) = \rho(g \cdot 1) = \rho(1)^g$ since ρ is a group homomorphism (we are using sum on the left hand side and multiplication on the right hand side as the group operations). Since by [1, Corollary 1.5.16. (p. 17)] we must have $\sum_{i=1}^r d_i^2 = |G|$ and as long as we take α to be a primitive root of unity (say $\alpha = e^{2\pi i/3}$), the representations we found are clearly pairwise not isomorphic, and we have that these are all we need. We can now divide Φ into the direct sum of three maps, namely:

$$\begin{array}{rccc} \Phi_{\alpha^i} & : & \mathbb{C} \otimes \mathbb{C} & \longrightarrow & \mathbb{C}[G] \\ & & x \otimes \xi & \longrightarrow & \sum_{k=0}^{2} \xi(1)\alpha^{ik} x 1_k \end{array}$$

for $i = 0, 1, 2$. Notice that to we used that the way $\xi \in \mathbb{C}$ acts on $x \in \mathbb{C}$ is by left multiplication by $\xi(1)$, since it is linear and one dimensional.

The above can be readily generalized to $\mathbb{Z}_m$. We can still find m one dimensional representations $\rho_i : G \longrightarrow \mathbb{C}^\times \cong \mathrm{GL}(\mathbb{C})$ for $i = 0, \dots, m-1$ given by $\rho_i(1) = \alpha^i$ and we still have that α must be an mth root of unity. Moreover [1, Corollary 1.5.16. (p. 17)] still applies and thus since $\sum_{i=1}^r d_i^2 = |G|$ is satisfied and taking α to be a primitive root of unity (say $\alpha = e^{2\pi i/n}$), the representations we found are still clearly pairwise not isomorphic, and we have that these are all we need. Hence we can divide Φ into the direct sum of m maps, namely:

$$\begin{array}{rccc} \Phi_{\alpha^i} & : & \mathbb{C} \otimes \mathbb{C} & \longrightarrow & \mathbb{C}[G] \\ & & x \otimes \xi & \longrightarrow & \sum_{k=0}^{m-1} \xi(1)\alpha^{ik} x 1_k \end{array}$$

for $i = 0, \dots, m-1$. We again used that the way $\xi \in \mathbb{C}$ acts on $x \in \mathbb{C}$ is by left multiplication by $\xi(1)$.

Exercise 1.7.8.

Show that $\mathrm{End}_{G\times G}(K[G])$ is $Z(K[G])$ (notice that the structure of $K[G]$ as representation of $G \times G$ is given by $\rho(g,g')(1_h) = 1_{g'hg^{-1}}$). For this, consider the map:

$$\begin{array}{ccccccccc} \phi & : & Z(K[G]) & \longrightarrow & \mathrm{End}_{G\times G}(K[G]) & & & & \\ & & r & \longmapsto & \phi_r & : & K[G] & \longrightarrow & K[G] \\ & & & & & & s & \longmapsto & sr \end{array}$$

we claim that it is a K algebra isomorphism. First, it is well defined: for every $\sum_{g\in G} r_g 1_g = r \in Z(K[G])$ we clearly have that ϕ_r is K linear since it is simply multiplication on the right, and to see that it is an intertwiner is enough to set $g, g', h \in G$ and check it on the basis elements:

$$\phi_r \circ \rho(g,g')(1_h) = \phi_r(1_{g'hg^{-1}}) = 1_{g'hg^{-1}} r = \sum_{f\in G} r_f 1_{g'h^{g^{-1}f}}$$

$$\rho(g,g') \circ \phi_r(1_h) = \rho(g,g')(1_h r) = \rho(g,g')(r1_h) = \rho(g,g')\left(\sum_{f\in G} r_f 1_{fh}\right) = \sum_{f\in G} r_f 1_{g'fhg^{-1}}$$

where we have used $r \in Z(K[G])$. We can re-index this to:

$$\begin{aligned} \phi_r \circ \rho(g,g')(1_h) &= \sum_{k\in G} r_{gh^{-1}(g')^{-1}k} 1_k \\ \rho(g,g') \circ \phi_r(1_h) &= \sum_{k\in G} r_{(g')^{-1}kgh^{-1}} 1_k \end{aligned}$$

and hence it is enough to see that $r_{gh^{-1}(g')^{-1}k} = r_{(g')^{-1}kgh^{-1}}$. For this, notice that since $r \in Z(K[G])$ we have (after re-indexing):

$$\begin{aligned} \sum_{k\in G} r_{gh^{-1}(g')^{-1}k} 1_k = \sum_{f\in G} r_f 1_{g'hg^{-1}f} &= 1_{g'hg^{-1}} r \\ &= r1_{g'hg^{-1}} = \sum_{f\in G} 1_{fg'hg^{-1}} = \sum_{k\in G} r_{kgh^{-1}(g')^{-1}} 1_k \end{aligned}$$

meaning that $r_{gh^{-1}(g')^{-1}k} = r_{kgh^{-1}(g')^{-1}}$ for all $g, h, g', k \in G$, so indeed $r_{gh^{-1}(g')^{-1}k} = r_{kgh^{-1}(g')^{-1}} = r_{(g')^{-1}kgh^{-1}}$ as desired.

Now that we know that ϕ is well defined, seeing that it is a K algebra homomorphism is straightforward, letting $k \in K$, $u, v \in Z(K[G])$, $s \in K[G]$ we have:

$$\begin{aligned} \phi(ku)(s) &= s(ku) = k(su) = k\phi(u)(s) \\ \phi(u+v)(s) &= s(u+v) = su + sv = \phi(u)(s) + \phi(v)(s) \\ \phi(uv)(s) &= s(uv) = s(vu) = \phi(u)(sv) = \phi(u) \circ \phi(v)(s) \end{aligned}$$

as desired. Moreover, it is injective since if we have $r \in Z(K[G])$ such that $\phi(r)(s) = s$ for all $s \in K[G]$, this in particular happens over the basis $\{1_g\}_{g\in G}$ of $K[G]$, and thus:

$$1_g = \phi(r)(1_g) = 1_g \sum_{h\in G} r_h 1_h = \sum_{h\in G} r_h 1_g h$$

and thus $r_e = 1$ and if $h \neq e$ we must have $r_h = 0$, meaning that $r = 1_e$ and we obtain injectivity. Finally, to prove that ϕ is also surjective, given $\psi \in \mathrm{End}_{G\times G}(K[G])$, we claim that $\sum_{g\in G} c_g 1_g = \psi(1_e) \in Z(K[G])$, and it is enough to check this on the basis $\{1_g\}_{g\in G}$ of $K[G]$:

$$\begin{aligned} 1_g\psi(1_e) &= 1_g \sum_{h\in G} c_h 1_h = \sum_{h\in G} c_h 1_{gh} = \rho(e,g)\left(\sum_{h\in G} c_h 1_h\right) = \rho(e,g)(\psi(1_e)) \\ &= \psi \circ \rho(e,g)(1_e) = \psi(1_g) = \psi \circ \rho(g^{-1},e)(1_e) = \rho(g^{-1},e)(\psi(1_e)) \\ &= \rho(g^{-1},e)\left(\sum_{h\in G} c_h 1_h\right) = \sum_{h\in G} c_h 1_{hg} = \left(\sum_{h\in G} c_h 1_h\right) 1_g = \psi(1_e) 1_g \end{aligned}$$

where we have abused that ψ is an intertwiner and ρ is linear. In particular, this chain of equalities gives us that $\rho(e,g)(\psi(1_e)) = 1_g\psi(1_e)$, a fact that we will use below. Equipped with this, we now check that $\phi(\psi(1_e)) = \psi$, which by linearity it is enough to verify it on the basis $\{1_g\}_{g\in G}$ of $K[G]$. Notice that:

$$\psi(1_g) = \psi(\rho(e,g)(1_e)) = \rho(e,g)(\psi(1_e)) = 1_g\psi(1_e) = \phi(\psi(1_e))(1_g)$$

and thus ϕ is indeed surjective.

We found a bijective K algebra homomorphism between $Z(K[G])$ and $\mathrm{End}_{G\times G}(K[G])$, and thus they are isomorphic.

Exercise 1.7.12.

Let K an algebraically closed field, with $\mathrm{char}(K)$ not dividing $|G|$, (ρ_i, V_i) for $i = 1, \dots, r$ be the representatives of the isomorphism classes of simple representations of a group G with irreducible characters $\chi_1, \dots, \chi_r$.

To see that they form a basis of $Z(K[G])$, we know that $\dim_K(Z(K[G])) = r$ by the proof of [1, Corollary 1.5.16. (p. 17)] and that they are linearly independent since by [1, Theorem 1.7.9. (p. 25)] we have $\chi_i = (|G|/\dim_K(V_i))\epsilon_i$ for $i = 1, \dots, r$ with ϵ_i the primitive central idempotents in $K[G]$, so the linear independence of $\{\epsilon_i\}i = 1^r$ (which is clear since they are in different direct summands) implies the linear independence of $\{\chi_i\}i = 1^r$. Hence to see that these form a basis it is enough to see that they lie in $Z(K[G]) = \{f : G \longrightarrow K | f(y) = f(x^{-1}yx) \forall x, y \in G\}$, which is the characterization of the center (given by the proof of [1, Corollary 1.5.16. (p. 17)]) that we want to use (since the natural way of seeing the χ_i, $i = 1, \dots, r$ is as functions). Let $x, y \in G$, then:

$$\begin{aligned}\chi_i(x^{-1}yx) &= \mathrm{Tr}(\rho_i(x^{-1}yx)) = \mathrm{Tr}(\rho_i(x^{-1})\rho_i(y)\rho_i(x)) \\ &= \mathrm{Tr}(\rho_i(y)\rho_i(x)\rho_i(x^{-1})) = \mathrm{Tr}(\rho_i(y)) = \chi_i(y)\end{aligned}$$

for all $i = 1, \dots, r$ since we are dealing with matrices and thus $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$ applies. Hence $\chi_i \in Z(K[G])$ for $i = 1, \dots, r$ and indeed $\{\chi_i\}i = 1^r$ form a basis.

Moreover, noticing as in class that $\dim_K(V) \neq 0$ in K, we can always divide by it and thus we have:

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\chi_j(g^{-1}h) &= \frac{1}{|G|}\sum_{g\in G}\frac{|G|}{\dim_K(V_i)}\epsilon_i(g)\frac{|G|}{\dim_K(V_j)}\epsilon_j(g^{-1}h) \\ &= \frac{|G|}{\dim_K(V_i)\dim_K(V_j)}\sum_{g\in G}\epsilon_i(g)\epsilon_j(g^{-1}h) \\ &= \frac{|G|}{\dim_K(V_i)\dim_K(V_j)}\epsilon_i * \epsilon_j(h) = \frac{|G|}{\dim_K(V_i)\dim_K(V_j)}\delta_{ij}\epsilon_i(h) \\ &= \frac{\delta_{ij}\chi_i(h)}{\dim_K(V_i)} = \frac{\delta_{ij}\chi_i(h)}{\chi_i(e)}\end{aligned}$$

since $\dim_K(V_i) = \chi_i(e)$ because on the right hand side we are adding the number of elements of the diagonal of the identity, which is exactly the dimension of the corresponding vector space.

Exercise 1.7.15.1.

We compute the character table of the finite cyclic groups $\mathbb{Z}_m$, $m \in \mathbb{N}$ over $\mathbb{C}$. We have by Exercise 1.7.4. that there are m isomorphism classes of (one dimensional) simple representations of $\mathbb{Z}_m$, given by ρ_{α^k} for $k = 0, \ldots, m-1$ and $\alpha = e^{2\pi i/n}$. Moreover, each element is it's own conjugacy class (because $\mathbb{Z}_m$ is cyclic) and thus we may index these by $0, \ldots, m-1$. Finally, notice that for $n \in \mathbb{Z}_m$ we have $\rho_{\alpha^k}(n) = \alpha^{kn}$ by definition, and thus the (compact) character table is:

	$\{n\}$
ρ_{α^k}	α^{kn}

This is just a compact form of the full sized table, the top row takes entries $\{n\}$ for $n \in \mathbb{Z}_m$ and the left column takes entries ρ_{α^k} for $k = 0, \ldots, m-1$, and we fill the table with the corresponding α^{kn}.

Exercise 1.7.16.

Let Let K an algebraically closed field, with $\text{char}(K)$ not dividing $|G|$, (ρ_i, V_i) for $i = 1, \ldots, r$ be the representatives of the isomorphism classes of simple representations of a group G with irreducible characters $\chi_1, \ldots, \chi_r$ and representatives of the conjugacy classes $g_1, \ldots, g_r$, denoted $C_{g_1}, \ldots, C_{g_r}$. With X the character table matrix, X' its transpose, Z the diagonal matrix of the cardinality of the centralizers and E the permutation matrix $E_{ij} = 1$ if g_i^{-1} lies in the conjugacy class of g_j.

We first prove that $XZ^{-1}EX' = 1$. Notice that Z^{-1} is just a diagonal filled with the inverses of the elements in the diagonal of Z. Thus by the definition of matrix multiplication:

$$(XZ^{-1})_{ij} = \sum_{k=1}^{r} X_{ik} Z^{-1}_{kj} = \sum_{k=1}^{r} \chi_i(g_k) \frac{\delta_{kj}}{|Z(g_j)|} = \frac{\chi_i(g_j)}{|Z(g_j)|}$$

and:

$$(EX')_{ij} = \sum_{k=1}^{r} E_{ij} X'_{kj} = \sum_{k=1}^{r} E_{ij} X_{jk} = \sum_{k=1}^{r} E_{ij} \chi_j(g_k)$$

hence:

$$\begin{aligned}
(XZ^{-1}EX')_{ij} &= \sum_{l=1}^{r} (XZ^{-1})_{il} (EX')_{lj} = \sum_{l=1}^{r} \frac{\chi_i(g_l)}{|Z(g_l)|} \sum_{k=1} E_{lk} \chi_j(g_k) \\
&= \sum_{l=1}^{r} \sum_{k=1}^{r} \frac{\chi_i(g_l)}{|Z(g_l)|} E_{lk} \chi_j(g_k) = \sum_{l=1}^{r} \frac{\chi_i(g_l)}{|Z(g_l)|} \chi_j(g_l^{-1}) \\
&= \frac{|C_{g_l}|}{|G|} \sum_{l=1}^{r} \chi_i(g_l) \chi_j(g_l^{-1}) = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g^{-1}) = \delta_{ij}
\end{aligned}$$

We used that E_{lk} is 1 if and only if g_l^{-1} lies in the conjugacy class of g_k, and in that case $\chi_j(g_k) = \chi_j(g_l^{-1})$ since the characters are constant in the conjugacy classes, that if $|C_{g_l}|$ is the cardinality of the conjugacy class of g_l, then $|C_{g_l}||Z(g_l)| = |G|$ by the Orbit-Stabilizer Theorem, that summing the value of the conjugacy classes and multiplying by their respective number of elements is the same thing as summing over all the elements in the group (since $\chi_i(g_l)\chi_j(g_l^{-1})$ is a value constant in the conjugacy class), and then we used Exercise 1.7.12. with $h = e$, the so called Schur's orthogonality relations. This indeed yields $XZ^{-1}EX' = 1$.

Moreover, we can check that $E^2 = 1$ by computing:

$$(EE)_{ij} = \sum_{k=1}^{r} E_{ij} E_{kj}$$

and for this to be different than 0 we need $E_{ik} = E_{kj} = 1$, and then on one hand g_i^{-1} lies in the conjugacy class of g_k and on the other hand g_k^{-1} lies in the conjugacy class of g_j. The second statement is equivalent to g_k lying in the conjugacy class of g_j^{-1} (by simply taking inverses on both sides). Hence we have that g_i^{-1} lies in the conjugacy class of g_j^{-1}, which is equivalent to g_i lying in the conjugacy class of g_j. Since these are representatives of the isomorphism classes, we must have that $i = j$. This yields $(EE)_{ij} = \sum_{k=1}^r E_{ij}E_{kj} = \delta_{ij}$ and thus indeed $E^2 = 1$.

Now, it follows that:

$$\begin{aligned} XZ^{-1}EX' = 1 &\iff Z^{-1}EX' = X^{-1} \iff EX' = ZX^{-1} \\ &\iff X' = EZX^{-1} \iff X'X = EZ \end{aligned}$$

by multiplying by the various inverses, keeping in mind that we proved $E^{-1} = E$. Thus this yields:

$$\begin{aligned} \sum_{k=1}^r \chi_k(g_i)\chi_k(g_j) &= \sum_{k=1}^r X_{ki}X_{kj} = \sum_{k=1}^r X'_{ik}X_{kj} = (X'X)_{ij} \\ &= (EZ)_{ij} = \sum_{k=1}^r E_{ik}Z_{kj} = E_{ij}Z_{jj} = E_{ij}|Z(g_j)| \end{aligned}$$

Notice now that g_j^{-1} must lie in only one of the conjugacy classes of g_k, since these are the representatives of the isomorphism classes. If g_j^{-1} lies in the conjugacy class of g_i, then the above becomes: $\sum_{k=1}^r \chi_k(g_j^{-1})\chi_k(g_j) = |Z(g_j)|$. If g_j^{-1} does not lie in the conjugacy class of g_i, this means that g_j^{-1} lies in the conjugacy class of g_l with $l \neq i$, and thus g_j lies in the conjugacy class of g_l^{-1}. Hence: $\sum_{k=1}^r \chi_k(g_i)\chi_k(g_l^{-1}) = \sum_{k=1}^r \chi_k(g_i)\chi_k(g_j) = E_{ij}|Z(g_j)| = 0$. Putting this last paragraph in a compact formula is:

$$\sum_{k=1}^r \chi_k(g_i)\chi_k(g_j^{-1}) = \begin{cases} |Z(g_j)| \text{ if } i = j, \\ 0 \text{ if } i \neq j, \end{cases}$$

the desired result.

Exercise 1.7.17.

Let Let K an algebraically closed field, with $\text{char}(K)$ not dividing $|G|$, (ρ_i, V_i) for $i = 1, \ldots, r$ be the representatives of the isomorphism classes of simple representations of a group G with irreducible characters $\chi_1, \ldots, \chi_r$ and representatives of the conjugacy classes $g_1, \ldots, g_r$, denoted $C_1, \ldots, C_r$.

We first show that $1_{C_j} = (|C_j|/|G|) \sum_{i=1}^r \chi_i(g_j^{-1})\chi_i$. For this, notice that if $g \in C_j$ then:

$$\frac{|C_j|}{|G|} \sum_{i=1}^r \chi_i(g_j^{-1})\chi_i(g) = \frac{|C_j|}{|G|} \sum_{i=1}^r \chi_i(g_j^{-1})\chi_i(g_j) = \frac{|C_j|}{|G|} |Z(g_j)| = 1$$

where we used that χ_i is constant in conjugacy classes for all $i = 1, \ldots, r$, Exercise 1.7.16. and the Orbit-Stabilizer Theorem. Employing these same tools, notice that if $g \notin C_j$ then $g \in C_k$ with $k \neq j$ and thus:

$$\frac{|C_j|}{|G|} \sum_{i=1}^r \chi_i(g_j^{-1})\chi_i(g) = \frac{|C_j|}{|G|} \sum_{i=1}^r \chi_i(g_j^{-1})\chi_i(g_k) = \frac{|C_j|}{|G|} 0 = 0.$$

Thus this expression behaves as the indicator, as desired.

We now show that for any $z \in G$ we have $|\{x \in C_j, y \in C_k : xy = z\}| = (|C_j||C_k|/|G|) \sum_{i=1}^r \chi_i(g_j^{-1})\chi_i(g_k^{-1})\chi_i(z)/\chi_i(1)$. For this, we simply compute:

$$1_{C_j} * 1_{C_k}(z) = \sum_{\substack{x,y \in G \\ xy=z}} 1_{C_j}(x) 1_{C_k}(y) = |\{x \in C_j, y \in C_k : xy = z\}|$$

since we are adding one for each instance that we have $x \in C_j$ and $y \in C_k$ with $xy = z$, precisely the desired cardinality. Moreover:

$$\begin{aligned}
1_{C_j} * 1_{C_k}(z) &= \sum_{\substack{x,y \in G \\ xy=z}} 1_{C_j}(x) 1_{C_k}(y) \\
&= \sum_{\substack{x,y \in G \\ xy=z}} \left(\frac{|C_j|}{|G|} \sum_{l=1}^r \chi_l(g_j^{-1})\chi_l(x) \right) \left(\frac{|C_k|}{|G|} \sum_{m=1}^r \chi_m(g_k^{-1})\chi_m(y) \right) \\
&= \frac{|C_j||C_k|}{|G||G|} \sum_{l=1}^r \sum_{m=1}^r \sum_{\substack{x,y \in G \\ xy=z}} \chi_l(g_j^{-1})\chi_l(x)\chi_m(g_k^{-1})\chi_m(y) \\
&= \frac{|C_j||C_k|}{|G||G|} \sum_{l=1}^r \sum_{m=1}^r \chi_l(g_j^{-1})\chi_m(g_k^{-1}) \sum_{\substack{x \in G \\ y=x^{-1}z}} \chi_l(x)\chi_m(x^{-1}z) \\
&= \frac{|C_j||C_k|}{|G|} \sum_{l=1}^r \sum_{m=1}^r \chi_l(g_j^{-1})\chi_m(g_k^{-1}) \frac{\delta_{lm}\chi_l(z)}{\chi_l(1)} \\
&= \frac{|C_j||C_k|}{|G|} \sum_{l=1}^r \frac{\chi_l(g_j^{-1})\chi_l(g_k^{-1})\chi_l(z)}{\chi_l(1)}
\end{aligned}$$

where we have used the expression computed above and Exercise 1.7.16. This gives the desired equality.

Exercise 1.8.8.

If χ is the character of a representation of a finite group in a complex vector space of dimension n, we check that then $|\chi(g)| \leq \chi(1)$ for all $g \in G$.

Since G is a finite group, we must have that all $g \in G$ have finite order, that is, $g^m = e$ for some non zero $m \in \mathbb{N}$. Then $\rho(g)^m = \rho(g^m) = \rho(e) = 1$. Hence since $\mathbb{C}$ is algebraically closed and we are in a representation of dimension n, the matrix $\rho(g)$ has n eigenvalues and thus $\mathrm{Tr}(\rho(g))$ is the sum of those eigenvalues. Let v be an eigenvector with eigenvalue λ, then $\lambda^m v = \rho(g)^m v = v$ meaning that $\lambda^m = 1$, and thus all the eigenvalues of $\rho(g)$ are a root of unity. In particular they have modulus 1 and:

$$|\chi(g)| = |\mathrm{Tr}(\rho(g))| = |\lambda_1 + \cdots + \lambda_n| \leq |\lambda_1| + \cdots + |\lambda_n| = 1 + \cdots + 1 = n = \chi(e)$$

where we used that since $\rho(e) = 1$, that has trace the dimension of the vector space, and thus $\chi(e) = n$.

We were instructed not to present a proof for the second part of the exercise since the statement may not be correct. However, one implication is true since if $g \in \ker(\rho)$ then $\rho(g) = 1$ and thus by the reasoning above $\chi(g) = n = \chi(e)$.

References

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[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.