Representations of Finite Groups - Homework 5

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Exercise 2.1.13.

For each $x \in X$, denote by 1_x the indicator at x. Show that $\rho_X(g)(1_x) = 1_{g \cdot x}$.

Let $y \in X$, then:

$$\rho_X(g)(1_x)(y) = 1_x(g^{-1} \cdot y) = \begin{cases} 1 \text{ if } x = g^{-1} \cdot y, \\ 0 \text{ if } x \neq g^{-1} \cdot y, \end{cases} = \begin{cases} 1 \text{ if } g \cdot x = y, \\ 0 \text{ if } g \cdot x \neq y, \end{cases} = 1_{g \cdot x}(y),$$

that is $\rho_X(g)(1_x) = 1_{g \cdot x}$, as desired.

Exercise 2.1.14.

Show that the character of a permutation representation is given by the number of fixed points, that is, $\text{Tr}(\rho_X(g)) = |X^g|$ where $X^g = \{x \in X : g \cdot x = x\}$.

Let $g \in G$, to compute the trace of $\rho_X(g)$ we consider the basis of $K[X]$ given by $\{1_x\}_{x \in X}$ (the indicators). They clearly generate since any $f \in K[X]$ can be written as $f = \sum_{x \in X} f(x) 1_x$ and they are linearly independent since they all take non zero values on different elements of X, and thus if $\sum_{x \in X} \alpha_x 1_x = 0$ for some $\{\alpha_x\}_{x \in X} \subset K$ then evaluating at each $x \in X$ we find that $\alpha_x = 0$, and thus linearly independence follows.

We want to see when do we have $\rho_X(g)(1_x) = \beta_x 1_x$, and thus these β_x for $x \in X$ will add up to the trace. By Exercise 2.1.13. we have that $\beta_x \in \{0, 1\}$ for $x \in X$, and $1_x = \rho_X(g)(1_x) = 1_{g \cdot x}$ if and only if $x = g \cdot x$ if and only if $x \in X^g$. Thus we add 1 for each such x, and we have as many x's as $|X^g|$, hence:

$$\text{Tr}(\rho_X(g)) = \sum_{x \in X} \beta_x = \sum_{x \in X^g} 1 = |X^g|$$

as desired.

Exercise 2.1.15.

If X and Y are isomorphic as G sets, prove that $K[X]$ and $K[Y]$ are isomorphic as representations of G.

Let $\phi : X \longrightarrow Y$ be the G set isomorphism, that is, for every $g \in G$ we have $\phi(g \cdot x) = g \cdot \phi(x)$. Consider:

$$\begin{array}{rcccc} T & : & K[X] & \longrightarrow & K[Y] \\ & & 1_x & \longrightarrow & 1_{\phi(x)} \end{array}$$

and extend linearly. This is linear by definition and bijective since:

$$\begin{array}{rcccc} S & : & K[Y] & \longrightarrow & K[X] \\ & & 1_y & \longrightarrow & 1_{\phi^{-1}(y)} \end{array}$$

and extend linearly, is its inverse: for all $x \in X$ and $y \in Y$:

$$S \circ T(1_x) = S(1_{\phi(x)}) = 1_{\phi^{-1}(\phi(x))} = 1_x$$
$$T \circ S(1_y) = S(1_{\phi^{-1}(y)}) = 1_{\phi(\phi^{-1}(y))} = 1_y$$

and thus $S = T^{-1}$ and T is a bijective linear map (by linearity, it is enough to check properties on the basis). Moreover, it is also an intertwiner since for all $g \in G$ and $x \in X$:

$$T \circ \rho_X(g)(1_x) = T(1_{g \cdot x}) = 1_{\phi(g \cdot x)} = 1_{g \cdot \phi(x)} = \rho_Y(g)(1_{\phi(x)}) = \rho_Y(g) \circ T(1_x),$$

and hence T is a bijective intertwiner, thus an isomorphism of representations.

Exercise 2.1.16.

Let X be a G set, and for each orbit $\mathcal{O} \subset X$ identify $K[\mathcal{O}]$ with the subspace of $K[X]$ consisting of functions supported on $\mathcal{O}$. Prove that $K[\mathcal{O}]$ is an invariant subspace and $K[X] = \oplus_{\mathcal{O}\in G\backslash X} K[\mathcal{O}]$.

We clearly have that $K[\mathcal{O}]$ is invariant, as if we set $\mathcal{O} = G_x$ for some representative $x \in X$, we have $K[G_x] = \langle 1_{g\cdot x}\rangle_{g\in G}$ as a K vector space (by definition). We can then write every $\xi \in K[G_x]$ as $\xi = \sum_{h\in G} \xi(h)1_{h\cdot x}$, and thus for every $g \in G$:

$$\rho_X(g)(\xi) = \sum_{h\in G} \xi(h)1_{(gh)\cdot x} = \sum_{k\in G} \xi(g^{-1}k)1_{k\cdot x} \in K[G_x]$$

via re-indexing. Hence $\rho_X(g)(K[G_x]) \subset K[G_x]$ for all $g \in G$, obtaining invariance of $K[G_x]$.

To prove the direct sum, notice that given $x \in X$, the indicator $1_x \in K[\mathcal{O}]$ for exactly one orbit: if $1_x \in K[G_y] \cap K[G_z]$ then there exist $g, h \in G$ with $x = g \cdot y$ and $x = h \cdot z$, meaning that $g \cdot y = h \cdot z$ which happens if and only if $z = (h^{-1}g) \cdot y$ and thus $G_y = G_z$. Hence, since $1_x \in K[G_x]$ for all $x \in X$, all indicators belong to at least one such $K[\mathcal{O}]$ and thus $K[X] = \cup_{\mathcal{O}\in G\backslash X} K[\mathcal{O}]$. Moreover, if $\xi \in K[G_x]$ we have that $\xi \notin K[G_y]$ whenever $G_y \neq G_x$, since otherwise the indicators forming ξ would be in $K[\mathcal{O}]$ for two different orbits. This means that for $G_x \neq G_y$ we have $K[G_x] \cap K[G_y] = \{0\}$, and this pairwise trivial intersection means that the union above is in fact a direct sum:

$$K[X] = \bigoplus_{\mathcal{O}\in G\backslash X} K[\mathcal{O}],$$

as desired.

Exercise 2.1.17.

Prove that the subspace $K[X]_0 = \{f : X \longrightarrow K | \sum_{x\in X} f(x) = 0\}$ is always an invariant subspace of $K[X]$. If $\text{char}(K)$ does not divide $|X|$, then it has an invariant complement.

We first check invariance. Let $f \in K[X]_0$, that is $f = \sum_{x\in X} f(x)1_x$ with $\sum_{x\in X} f(x) = 0$, and $g \in G$. Then:

$$\rho_X(g)(f) = \sum_{x\in X} f(x)1_{g\cdot x} = \sum_{y\in X} f(g^{-1}\cdot y)1_y$$

via re-indexing, so:

$$\sum_{x\in X} \rho_X(g)(f)(x) = \sum_{x\in X} f(g^{-1}\cdot x) = \sum_{y\in X} f(y) = 0,$$

again via re-indexing (notice that $g^{-1}\cdot x = y$ if and only if $x = g\cdot y$ and multiplying by g is a bijective map from X ot X, so we still sum over all elements in X when re-indexing). Thus $\rho_X(g)(f) \in K[X]_0$.

We now fix $x_0 \in X$ and notice that $K[X]_0$ has $\{1_{x_0} - 1_x\}_{x\in X}$ as a basis. For this, we clearly have $\{1_{x_0} - 1_x\}_{x\in X} \subset K[X]_0$ since $\sum_{y\in X}(1_{x_0} - 1_x)(y) = 1_{x_0}(x_0) - 1_x(x) = 0$ for all $x_0 \neq x \in X$, and this set is linearly independent: suppose that we have $\alpha_x \in K$ for $x_0 \neq x \in X$ with:

$$0 = \sum_{x_0\neq x\in X} \alpha_x(1_{x_0} - 1_x) = \sum_{x_0\neq x\in X} \alpha_x 1_{x_0} - \sum_{x_0\neq x\in X} \alpha_x 1_x$$

and since $\{1_x\}_{x\in X}$ is a basis of $K[X]$, we must have $\alpha_x = 0$ for $x_0 \neq x \in X$, obtaining the desired linear independence. This means that $\dim_K(K[X]_0) \geq |X| - 1$. Finally, since $\dim_K(K[X]) = |X|$ and $\{1_x\}_{x\in X} \not\subset K[X]_0$ since $\sum_{y\in X} 1_x(y) = 1$ for all $x \in X$, we have that $K[X]_0 \subsetneq K[X]$ and thus $\dim_K(K[X]_0) < \dim_K(K[X])$, meaning that $\dim_K(K[X]_0) = |X| - 1$. Since $|\{1_{x_0} - 1_x\}_{x\in X}| = |X| - 1$, it follows that $\{1_{x_0} - 1_x\}_{x\in X}$ is a basis (linearly independent set in $K[X]_0$ of the same cardinality as its dimension).

We now assume that $\text{char}(K)$ does not divide $|X|$. Notice that W the space of constant functions is an invariant subspace of dimension 1. First, notice that if $f \in W$ then $f(x) = \alpha \in K$ for all $x \in X$, and thus $f = \sum_{x\in X} f(x)1_x = \sum_{x\in X} \alpha 1_x = \alpha\sum_{x\in X} 1_x$, and vice-versa if $\alpha \in K$ then $\alpha\sum_{x\in X} 1_x = \sum_{x\in X} \alpha 1_x \in W$. Thus $\{\sum_{x\in X} 1_x\}$ is a basis of W, so $\dim_K(W) = 1$. Moreover, W is invariant since for any $f \in W$, say $f(x) = \alpha$ for all $x \in X$, and $g \in G$, we have $\rho_X(g)(f)(x) = f(g^{-1}\cdot x) = \alpha$ for all $x \in X$ and thus $\rho_X(g)(f) \in W$. Finally, we claim that W is the invariant complement of $K[X]_0$, and start by noticing $K[X]_0 \cap W = \{0\}$: given $f \in W$ with $f(x) = \alpha \in K$ for all $x \in X$ we have for any $g \in G$:

$$\sum_{x\in X} f(x) = \sum_{x\in X} \alpha = \alpha|X|.$$

Thus since $\text{char}(K)$ does not divide $|X|$ we have that $f \in K[X]_0$ if and only if $\alpha = 0$ and thus $f = 0$. Thus $K[X]_0$ and W are in direct sum. Moreover $\dim_K(K[X]_0 \oplus W) = \dim_K(K[X]_0) + \dim_K(W) = |X|$ so $K[X]_0 \oplus W = K[X]$ and W is indeed the invariant complement of $K[X]_0$, as desired.

Exercise 2.2.7.

List the six permutations of $\mathbb{S}_3$ in lexicographic order and draw the multiplication table.

We usually denote the elements of $\mathbb{S}_3$ as id, (12), (23), (13), (123), (132). In this same order but in permutation string notation, they are 123, 213, 132, 321, 231, 312. Now it is easy to order them according to the lexicographic order:

$$123 < 132 < 213 < 231 < 312 < 321.$$

The multiplication table is:

	id	(12)	(23)	(13)	(123)	(132)
id	id	(12)	(23)	(13)	(123)	(132)
(12)	(12)	id	(123)	(132)	(23)	(13)
(23)	(23)	(132)	id	(123)	(13)	(12)
(13)	(13)	(123)	(132)	id	(12)	(23)
(123)	(123)	(13)	(12)	(23)	(132)	id
(132)	(132)	(23)	(13)	(12)	id	(123)

Exercise 2.2.11.

Show that the number of permutations in $\mathbb{S}_n$ with cycle type (n) is $(n-1)!$.

Note that we have n positions to fill with n numbers, this can be done in $n!$ different ways. However, since we are considering permutations, we counted each one n times: it does not matter where we put the parenthesis, if two permutations have the elements in the same order, they are the same. Hence fixing a permutation of cycle type n, by choosing a different element each time to where to put the parenthesis, we note that we counted it n times instead of just one. Thus we have:

$$\frac{n!}{n} = (n-1)!$$

the number of permutations in $\mathbb{S}_n$ with cycle type (n).

Exercise 2.2.12.

Show that the order of an element of $\mathbb{S}_n$ whose cycle type is $\lambda = (\lambda_1, \ldots, \lambda_l)$ is the least common multiple of $\lambda_1, \ldots, \lambda_l$.

First, note that if $\sigma \in \mathbb{S}_n$ has cycle type λ, then by definition it can be decomposed in l disjoint cycles (in particular, they commute with each other) as $\sigma = \sigma_1 \cdots \sigma_l$, where σ_i has lenght λ_i (so in particular σ_i has order λ_i) for $i = 1, \ldots, l$. Now the order of σ is the minimum $r \in \mathbb{N}$ such that $\sigma^r = \mathrm{id}$, in which case:

$$\mathrm{id} = \sigma^r = \sigma_1^r \cdots \sigma_l^r$$

and by the uniqueness of the cycle decomposition, we must have $\sigma_i^r = \mathrm{id}$ meaning that λ_i divides r for $i = 1, \ldots, l$. Thus $r \in \mathbb{N}$ is the minimum such that it is a multiple of $\lambda_1, \ldots, \lambda_l$, hence by definition $r = \mathrm{lcm}(\lambda_1, \ldots, \lambda_l)$, as desired.

References

[1] A. Prasad, *Representation Theory: A Combinatorial Viewpoint*, Cambridge studies in advanced mathematics, 2015.

[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.