Representations of Finite Groups - Homework 6

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Exercise 2.4.2.

Given three finite sets X, Y, Z and kernels $k_1 \in K[X \times Y]$, $k_2 \in K[Y \times Z]$, prove that $T_{k_1} \circ T_{k_2} = T_{k_1 * k_2}$ where $k_1 * k_2 : X \times Z \longrightarrow K$ is defined by $k_1 * k_2(x, z) = \sum_{y \in Y} k_1(x, y) k_2(y, z)$ for all $(x, z) \in X \times Z$.

We first note that this composition makes sense since $T_{k_1} : K[Y] \longrightarrow K[X]$ and $T_{k_2} : K[Z] \longrightarrow K[Y]$, and moreover $T_{k_1 * k_2} : K[Z] \longrightarrow K[X]$, so the equality can actually be true. Now, consider any $f \in K[Z]$, we prove that $T_{k_1} \circ T_{k_2}(f) = T_{k_1 * k_2}(f)$. For this, let $x \in X$, we have:

$$\begin{aligned} T_{k_1} \circ T_{k_2}(f)(x) &= \sum_{y \in Y} k_1(x, y) T_{k_2}(f)(y) = \sum_{y \in Y} k_1(x, y) \sum_{z \in Z} k_2(y, z) f(z) \\ &= \sum_{y \in Y} \sum_{z \in Z} k_1(x, y) k_2(y, z) f(z) = \sum_{z \in Z} \sum_{y \in Y} k_1(x, y) k_2(y, z) f(z) \\ &= \sum_{z \in Z} k_1 * k_2(x, z) f(z) = T_{k_1 * k_2}(f)(x) \end{aligned}$$

proving the desired equality.

Exercise 2.4.5.

Show that $\dim_K(\mathrm{Hom}_G(K[X], 1)) = |G\backslash X|$, where 1 denotes the trivial representation of G. When K is algebraically closed and $\mathrm{char}(K)$ does not divide $|G|$, conclude that the multiplicity of 1 in $K[X]$ is the same as the number of G-orbits in X.

Notice that if we take $Y = \{*\}$ a singleton, then Y is a G set under the trivial action (the only possible one). Now the permutation representation $\rho_Y : G \longrightarrow \mathrm{GL}(K[Y])$ is exactly the trivial representation, since $K[Y]$ is one dimensional because $|Y| = 1$ and for any $f \in K[Y]$ we have that $\rho_Y(g)(f)(*) = f(g^{-1} \cdot *) = f(*)$ and thus $\rho_Y(g) = \mathrm{id}_{K[Y]}$. Applying now [1, Theorem 2.4.4 (p. 42)] we have that:

$$\dim_K(\mathrm{Hom}_G(K[X], 1)) = \dim_K(\mathrm{Hom}_G(K[X], K[\{*\}])) = |G\backslash X \times \{*\}| = |G\backslash X|$$

the desired result.

For the second part, notice that if $\mathrm{char}(K)$ does not divide $|G|$ then by [1, Theorem 1.4.3 (p. 13)] we have that $K[X]$ is completely reducible, and if K is algebraically closed then by [1, Exercise 1.3.11. (p. 10)], keeping in mind [1, Theorem 1.3.5 (p. 9)] and thus we may permute the elements inside Hom to count multiplicity, we have that $\dim_K(\mathrm{Hom}_G(K[X], 1))$ is the multiplicity of 1 in $K[X]$. Since that is $|G\backslash X|$ and $G\backslash X$ is the set of G-orbits of X, we have that the multiplicity is exactly the number of G-orbits in X, as desired.

Exercise 2.4.6.

Let X a finite G-set, show that $|G||G\backslash| = \sum_{g\in G} |X^g|$ (notice that rewriting like this is fine since $|G|$ is finite so dividing by it is always well defined).

We will prove this using the representation theory here developed, so in particular we need K algebraically closed and $\mathrm{char}(K)$ not dividing $|G|$. In that case, we can apply [1, Theorem 1.7.14 (p. 27)], [1, Exercise 2.1.14 (p. 34)] and [1, Exercise 2.4.5. (p. 42)].

Combining the first and the last in the context of the representations $(\rho_X, K[X])$ and $(\rho_Y, K[Y])$ as above, we have that:

$$|G\backslash X| = \dim_K(\mathrm{Hom}_G(K[X], 1)) = m = \langle \chi_{\rho_Y}, \chi_{\rho_X}\rangle_G = \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_Y}(g)\chi_{\rho_X}(g^{-1})$$
$$= \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_Y}(g^{-1})\chi_{\rho_X}(g) = \frac{1}{|G|}\sum_{g\in G} |X^g|$$

where we have re-indexed the sum to simplify our lives, noted that since $K[Y]$ is the trivial representation (and has dimension one) all the traces are taken over the identity, and thus all the characters have value 1, and we used the second reference above to compute the character of the permutation representation $K[X]$. This is the desired result.

Exercise 2.5.5.

Show that $x \in X_k$ is fixed by an element $g \in S_n$ if and only if x is a union of cycles of g. Use this to show that for each partition λ of n the number of elements of X_k fixed by an element of S_n with cycle decomposition λ is the number of ways of adding some parts of λ to get k.

$\Leftarrow$) Let $x \in X_k$ being a union of cycles of g, say $x = \cup_{j\in J}(c_{1_j}, \ldots, c_{m_j})$ where $(c_{1_j}, \ldots, c_{m_j})$ is a cycle of g for all $j \in J$, in the usual cycle notation. Then:

$$\begin{aligned} g{\cdot}x = g \cdot \bigcup_{j\in J}(c_{1_j}, \ldots, c_{m_j}) &= \bigcup_{j\in J}(g(c_{1_j}), \ldots, g(c_{m_j})) \\ = \bigcup_{j\in J}(c_{2_j}, \ldots, c_{m_j}, c_{i_j}) &= \bigcup_{j\in J}(c_{1_j}, \ldots, c_{m_j}) = x \end{aligned}$$

and thus x is fixed by g.

$\Rightarrow$) Suppose that x is not a union of cycles of g. This means that there is some $c \in x$ and $m \in \mathbb{N}^+$ such that $g^m(c) \notin x$. Notice that taking such m to be minimal, if $m > 1$ then $g^{m-1}(c) \in x$ and $g(g^{m-1}(c)) \notin x$, so by considering $g^{m-1}(c)$ we may always assume that $m = 1$. Hence, we found some $c \in x$ such that $g(x) \notin x$, meaning that x is not fixed by g. By contrapositive, we obtain the desired result.

Given λ a partition of n and $g \in S_n$ with cycle decomposition λ, the number of elements of X_k fixed by g is by the above the number of $x \in X_k$ such that x is a union of cycles of g. Thus since x must have cardinality k, to get one such x we consider cycles of g that add up to k elements in total, and we have to count how many such combinations we have. Since the dimension of the cycles of g is given by λ, the previous line is exactly the number of ways of adding parts of λ to get k, what we desired.

Exercise 2.5.6.

Let K be an algebraically closed field of characteristic greater than 3. Compute the character of the simple representation V_1 of S_3.

Consider the permutation representation $\rho : G \longrightarrow \mathrm{GL}(K[X_1])$, we have that $K[X_1] = V_0 \oplus V_1$. Notice that $V_0 = K[X_0] = K[\{*\}]$ and thus as seen above we have that V_0 is the trivial representation. Keeping in mind that both are simple, we obtain that for any $g \in S_n$:

$$\rho(g) = \begin{bmatrix} \rho|_{V_0}(g) & 0 \\ 0 & \rho|_{V_1}(g) \end{bmatrix}$$

as a square matrix. Hence:

$$\chi_{K[X_1]}(g) = \mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\rho|_{V_0}(g)) + \mathrm{Tr}(\rho|_{V_1}(g)).$$

Moreover, in virtue of [1, Exercise 2.1.14 (p. 34)] we can compute traces over $K[X_1]$, and since V_0 is the trivial representation we are taking traces over the identity in a one dimensional space, so the above becomes:

$$|X_1^g| = 1 + \mathrm{Tr}(\rho|_{V_1}(g)) \Longrightarrow \chi_{V_1}(g) = \mathrm{Tr}(\rho|_{V_1}(g)) = |X_1^g| - 1.$$

In virtue of [1, Theorem 2.2.15 (p. 37)] we have that the conjugacy classes in S_3 are given by the number of partitions of 3, namely $\lambda_1 = (1,1,1)$, $\lambda_2 = (2,1)$, $\lambda_3 = (3)$. We can consider id_{S_3}, (12), (123) as representatives for them, respectively. Notice that $X_1 = \{\{0\},\{1\},\{2\}\} \equiv [n]$ and thus $|X_1^{\mathrm{id}}| = 3$, $|X_1^{(12)}| = 1$, $|X_1^{(123)}| = 0$, meaning that $\chi_{V_1}(\mathrm{id}) = 2$, $\chi_{V_1}(12) = 0$, $\chi_{V_1}(123) = -1$ and the character table is:

	λ_1	λ_2	λ_3
$\rho\|_{V_1}$	2	0	-1

.

Exercise 2.5.7.

Show that the character value of the representation V_1 of S_n at a permutation with cycle type λ is $m_1(\lambda) - 1$ where $m_1(\lambda)$ is the number of times that 1 occurs in λ.

The reasoning above still holds (for the most part, until we work specifically for $n = 3$), namely given $g \in S_n$ we still have:

$$\chi_{V_1}(g) = |X_1^g| - 1.$$

Now, let g have cycle type λ, notice that since $X_1 \equiv [n]$ we have that $|X_1^g|$ is exactly the number of points fixed by g. Since λ gives the lengths of the cycles forming g, a 1 in λ corresponds bijectively to a fixed point by g (that is, for every point g fixes we must have exactly one 1 in λ), and thus $m_1(\lambda)$, the number of 1 in λ, is the number of fixed points by g. Hence $m_1(\lambda) = |X_1^g|$ and thus:

$$\chi_{V_1}(g) = m_1(\lambda) - 1$$

as desired.

References

[1] A. Prasad, *Representation Theory: A Combinatorial Viewpoint*, Cambridge studies in advanced mathematics, 2015.

[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.