Representations of Finite Groups - Homework 7

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Exercise 3.1.4.

Suppose that $\lambda = (n-l, l)$ and $\mu = (n-m, m)$ for $0 \leq l, m \leq \lfloor n/2 \rfloor$. Show that there exists a unique SSYT of shape μ and type λ if and only if $m \leq l$, and if $m > l$ then there is no SSYT of shape μ and type λ.

$\Rightarrow$) Suppose we have a unique SSYT of shape μ and type λ. This means that the first row has $n-m$ squares and the second has m squares, and they are filled with $n-l$ ones and l twos. Notice that we need the twos to fill the second row, otherwise we have a one in the second row and we violate the fact that a SSYT is, from top to bottom, strictly increasing in columns. This means that $l \geq m$.

$\Leftarrow$) Consider an empty Young diagram of shape μ. If $m \leq l$ then we have to fill the second row with twos (to satisfy that a SSYT is, from top to bottom, strictly increasing in columns), and we have to put the remaining twos in the first row starting from the right and fill the remaining space with ones (to satisfy that a SSYT is, from left to right, weakly increasing in columns). This is a SSYT of shape μ and type λ, and is the only way to do it to satisfy both conditions, obtaining existence and uniqueness. If $m > l$, as reasoned above, we will need to have a one in the second row and we violate the fact that a SSYT is, from top to bottom, strictly increasing in columns, and hence no SSYT of shape μ and type λ will exist.

Exercise 3.1.10.

Let $\lambda = (m, 1^k)$ for some $m, k \in \mathbb{N}^+$. Which are the partitions $\mu \leq \lambda$?

Consider $\mu = (\mu_1, \ldots, \mu_i)$ and $\lambda = (\lambda_1, \ldots, \lambda_j)$, where $j = k + 1$ (although this is irrelevant for the following discussion). Notice that if $\mu \leq \lambda$ then the first condition is that $\mu_1 \geq \lambda_1 = m$. Once this is satisfied, since μ is a partition, we must have that $\mu_l > 0$ for $0 \leq l \leq i$, and thus for every $r \leq \min(i, j)$ we automatically have:

$$\mu_1 + \mu_2 + \cdots + \mu_r \geq m + 1 + \cdots 1 = \lambda_1 + \lambda_2 + \cdots + \lambda_r$$

and hence $\mu \leq \lambda$. Thus the sole condition for a partition μ to be smaller or equal to λ is that $\mu_1 \geq m$.

Exercise 3.1.11.

Show that if μ and λ are partitions of $n \in \mathbb{N}$ with $\mu \leq \lambda$, then the number of parts in μ is at most the number of parts in λ.

Consider $\mu = (\mu_1, \dots, \mu_i)$ and $\lambda = (\lambda_1, \dots, \lambda_j)$, we want to see that if $\mu \leq \lambda$ then $i \leq j$. Since $i, j \in \mathbb{N}^+$, we can compare them and thus there are three options: $i < j$, $i = j$, $i > j$. Suppose that $i > j$, then we have that $\mu_{j+1} > 0$ and thus:

$$\mu_1 + \cdots + \mu_j < n = \lambda_1 + \cdots + \lambda_j$$

meaning that $\mu \not\leq \lambda$, a contradiction with the hypothesis. Hence only the other two are possible and we must have $i \leq j$ as desired.

Exercise 3.2.7.

Find the 5×4 matrix A to which VRSK would associate the SSYT's:

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \quad Q = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}.$$

In the following, we denote first the pair of boxes that we are taking and the shadow path that they form in the 5×4 matrix (with an explanation if needed), then we add a 1 in the outer edges of the shadow path. We begin with the lower right box and move left:

$$\left(\boxed{4}, \boxed{5}\right) \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left(\boxed{2}, \boxed{2}\right) \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{End} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We finished the row, so we use this final matrix as the (to be) shadow matrix, we move up one row and start the process again:

$$\left(\boxed{3}, \boxed{4}\right) \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{0} \end{bmatrix} \qquad \left(\boxed{2}, \boxed{3}\right) \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Notice that in the first step we need to have $(4, 5)$ as a shadow point, otherwise the shadow of the matrix will cover it and it will never again have the opportunity of being a shadow point. Continuing:

$$\left(\boxed{1}, \boxed{1}\right) \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \left(\boxed{1}, \boxed{1}\right) \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Where again in the third step we need to have $(2, 2)$ as a shadow point, otherwise the shadow of the matrix will cover it and it will never again have the opportunity of being a shadow point. This ends the algorithm.

The remaining matrix is:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Exercise 3.2.9.

Show that if $\text{RSK}(A) = (P, Q)$ then $\text{RSK}(A^T) = (Q, P)$.

Given a matrix A, taking the transpose means that $A^T_{ij} = A_{ji}$. Notice that when we draw a shadow path in A, all the relevant outer points defining it will still be the relevant outer points when we transpose A, and the transpose of the shadow path we obtained is exactly the corresponding shadow path of A^T. That is, we can draw a parallel of the application of RSK to A^T and to A: the shadow path of A^T in a given step, is simply the transpose of the shadow path of A in the corresponding step, and similarly the shadow matrix of A^T in any given step will be the transpose of the shadow matrix of A in that same step.

Set $\text{RSK}(A^T) = (P', Q')$. The above means that what is stored in P' in each step (the information on the columns of A^T) is in fact given by the rows of A in each step, which is stored in Q, and thus $P' = Q$. Similarly, what is stored in Q' in each step (the information on the rows of A^T) is in fact given by the columns of A in each step, which is stored in P, and thus $Q' = P$. This proves the desired result.

Exercise 3.2.12.

Show that the number of involutions in S_n is equal to the number of SYT of size n.

First, note that there is a bijection between elements in S_n and $n \times n$ permutation matrices. To see this, we introduce the following notation: let e_i for $0 \leq i \leq n$ be the $n \times 1$ column vector having 1 in the position i and 0 elsewhere, notice that $1_{n\times n} = [e_1 \cdots e_n]$. Now given $\sigma \in S_n$ we associate the matrix $[e_{\sigma(i)} \cdots e_{\sigma(n)}]$. This is a bijection since by definition the permutation matrices are the matrices obtaining by permuting the rows of a $n \times n$ identity matrix according to some permutation of the numbers $\{1, \dots, n\}$. Set P_n the permutation matrices of size n, the above shows that the number of involutions in S_n is equal to the number of involutions in P_n.

Second, notice that $A \in P_n$ is an involution when $A^2 = 1_{n\times n}$, so in particular $A = A^{-1}$. We now check that the inverse of a permutation matrix is its transpose:

$$(PP^T)_{ij} = \sum_{k=1}^{n} P_{ik}P^T_{kj} = \sum_{k=1}^{n} P_{ik}P_{jk} = \delta_{ij} = (1_{n\times n})_{ij}$$
$$(P^TP)_{ij} = \sum_{k=1}^{n} P^T_{ik}P_{kj} = \sum_{k=1}^{n} P_{ki}P_{kj} = \delta_{ij} = (1_{n\times n})_{ij}$$

since if $P_{ik} = 1$ then $P_{jk} = 0$ for $j \neq i$ since a permutation matrix has a single 1 per row, and if $P_{ki} = 1$ then $P_{kj} = 0$ for $j \neq i$ since a permutation matrix has a single 1 per column. This proves that $A^{-1} = A^T$. Thus this means that $A = A^{-1} = A^T$.

Third, applying [1, Theorem 3.2.11 (p. 67)] we have that RSK determines a bijection between the permutation matrices P_n and the set of pairs of SYT of the same shape. Since a permutation matrix in P_n is a $(1^n) \times (1^n)$ matrix, by [1, Theorem 3.2.2 (p. 63)] we have that the set of pairs of SYT have both size n by definition. For the subset of involutions of P_n we can apply Exercise 3.2.9. to obtain that $(P, Q) = \mathrm{RSK}(A) = \mathrm{RSK}(A^T) = (Q, P)$, and thus $P = Q$ so $\mathrm{RSK}(A) = (P, P)$. Notice how here we do not have a "pair" of SYT both of size n, but just one SYT of size n, namely there is a bijection between the set $\{(P, P) : P \text{ is a SYT of size } n\}$ and the set $\{P : P \text{ is a SYT of size } n\}$. Since RSK is still a bijection, the number of involutions in P_n is equal to the cardinality of $\{P : P \text{ is a SYT of size } n\}$.

Hence:

$$|\{\sigma \in S_n : \sigma^{-1} = \sigma\}| = |\{A \in P_n : A = A^T\}| = |\{P : P \text{ is a SYT of size } n\}|$$

that is, the number of involutions of S_n is the number of involutions of P_n which is the number of SYT of size n, which is what we wanted to prove.

Exercise on applying the VRSK Algorithm

Let:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

run the RSK algorithm using it as input.

Applying the RSK algorithm, we denote by brackets the updated original matrix and we denote by parenthesis the updated shadow matrix:

Step 1: $p = 1$ $\quad q = 1$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$p = 11$ $\quad q = 12$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$p = 111$ $\quad q = 122$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$p = 1112 \qquad\qquad q = 1223$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&2&0\\ 0&0&0&1&0&0&0\\ 0&1&2&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$p = 11124 \qquad\qquad q = 12233$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&2&0\\ 0&0&0&1&0&0&0\\ 0&1&2&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$p = 111247 \qquad\qquad q = 122336$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&2&0\\ 0&0&0&1&0&0&0\\ 0&1&2&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

Now we consider the shadow matrix as the original one, and we repeat the algorithm:

Step 2: $p = 2 \qquad\qquad q = 3$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&2&0\\ 0&0&0&1&0&0&0\\ 0&1&2&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$p = 23 \qquad\qquad q = 33$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&2&0\\ 0&0&0&1&0&0&0\\ 0&0&2&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$p = 233 \qquad\qquad q = 333$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&1&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$p = 2336 \qquad\qquad q = 3335$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&1&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

Updating again the original matrix by the shadow one, we repeat the algorithm:

Step 3:

$$p = 3 \qquad\qquad q = 4$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&1&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$p = 34 \qquad\qquad q = 45$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$p = 346 \qquad\qquad q = 455$$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

And updating original and shadow matrices once more, we repeat the algorithm for the last time:

Step 4: $p = 6$, $q = 5$

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

The remaining SSYT are:

$$P = \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&2&4&7\\ \hline 2&3&3&6 \\ \cline{1-4} 3&4&6 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}, \quad Q = \begin{array}{|c|c|c|c|c|c|} \hline 1&2&2&3&3&6\\ \hline 3&3&3&5 \\ \cline{1-4} 4&5&5 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}.$$

References

[1] A. Prasad, *Representation Theory: A Combinatorial Viewpoint*, Cambridge studies in advanced mathematics, 2015.

[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.