Representations of Finite Groups - Homework 8

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Exercise 4.2.2.

Let (ρ, V) be a representation of a group G over a field K and $\chi : G \longrightarrow K^*$ a multiplicative character. We prove that if ρ is simple, then $\rho \otimes \chi$ is simple.

We proceed by contrapositive: suppose $\rho \otimes \chi$ is not simple, that is, there exists a linear subspace $W \subseteq V$ such that $\rho \otimes \chi(g)(W) \subseteq W$ for all $g \in G$. Now for every $g \in G$:

$$\chi(g)\rho(g)(W) = \rho \otimes \chi(g)(W) \subseteq W \quad \Longrightarrow \quad \rho(g)(W) \subseteq \chi(g)^{-1}W \subseteq W$$

since $\chi(g) \in K^*$ is invertible and W is a linear subspace. Thus ρ is not simple, proving the contrapositive and thus the original claim, as desired.

Exercise 4.3.3.

Let m and n be arbitrary positive integers. Show that the number of partitions on n with parts bounded by m is equal to the number of partitions of n with at most m parts.

Suppose $\lambda = (\lambda_1, \ldots, \lambda_l)$, where we may change l. We consider the sets $A = \{\lambda \vdash n : \lambda_i \leq m \text{ for } i = 1, \ldots, l\}$ and $B = \{\lambda \vdash n : l \leq m\}$. We will now construct a bijection between them.

Consider the map (as sets):

$$\begin{array}{rcccc} R & : & A & \longrightarrow & B \\ & & \lambda & \longmapsto & \lambda' \end{array}$$

where λ' is the conjugate partition of λ. This is well defined; if λ is a partition of n with parts bounded by m, then $\lambda_1 \leq m$ is the length of λ' (this is immediate using that the Young diagrams of λ' is the transpose of the Young diagram of λ, since the length of λ' is the number of rows of its Young diagram which is λ_1 by construction) and clearly λ' is still a partition of n (again, immediate using the Young diagrams). Thus $\lambda' \in B$.

Consider the map (as sets):

$$\begin{array}{rcccc} S & : & B & \longrightarrow & A \\ & & \lambda & \longmapsto & \lambda' \end{array}$$

where λ' is the conjugate partition of λ. This is well defined; if λ is a partition of n with at most m parts, then $\lambda'_1 \leq m$ (we proved above that λ'_1 is the length of λ) so since it is true for the first part and these are in increasing order, all parts of λ' are bounded by m and clearly λ' is still a partition of n (as above). Thus $\lambda' \in A$.

Since transposing twice a Young diagram is itself, we have that $S \circ R(\lambda) = \lambda$ for all $\lambda \in A$ and $R \circ S(\lambda) = \lambda$ for all $\lambda \in B$, and thus $|A| = |B|$, the desired result.

Exercise 4.3.4.

For each partition λ of n, show that $f_\lambda = f_{\lambda'}$. Thus we want to see that the number of SYT of shape λ is the same as the number of SYT of shape λ'.

Notice that given Y a SYT of shape λ, this is in fact a SSYT of shape λ and type (1^n), so each of the integers in $\{1, \dots, n\}$ appears exactly once in Y, making the rows of Y strictly increasing (it already has columns strictly increasing since it is a SSYT). Thus P', the Young tableau obtained by transposing P (including the numbers that fill P), is of shape λ' by construction and it has the numbers $\{1, \dots, n\}$ appearing exactly one on it, strictly increasing in both rows and columns, so P' is a SYT of shape λ'.

This proves that the following is a well defined map:

$$\begin{array}{rcccc} R & : & \{\text{SYT of shape } \lambda\} & \longrightarrow & \{\text{SYT of shape } \lambda'\} \\ & & P & \longmapsto & P' \end{array}$$

and since $(\lambda')' = \lambda$, the same argument also proves that:

$$\begin{array}{rcccc} S & : & \{\text{SYT of shape } \lambda'\} & \longrightarrow & \{\text{SYT of shape } \lambda\} \\ & & P & \longmapsto & P' \end{array}$$

is a well defined map. Moreover since transposing twice does not change the SYT, we obtain $S \circ R(P) = P$ for all $P \in \{\text{SYT of shape } \lambda\}$ and $R \circ S(P) = P$ for all $P \in \{\text{SYT of shape } \lambda'\}$, and thus $f_\lambda = |\{\text{SYT of shape } \lambda\}| = |\{\text{SYT of shape } \lambda'\}| = f_{\lambda'}$, the desired result.

Exercise 5.1.3.

Express the product $m_{(2,1)}m_{(1,1)}$ as a linear combination of monomial symmetric functions.

For this, notice that since the final product will be symmetric, we only need to find the coefficients of a single monomial of the possible combinations that can arise when we are multiplying exponents given by the partitions $(2,1)$ and $(1,1)$. The possible monomials that we can have must come from a multi-index with shape a partition of 5, obtained by combining $(2,1)$ and $(1,1)$, and with at most 4 non zero entries. This tells us two things; one that $m_{(2,1)}m_{(1,1)}$ will be a sum of monomial symmetric functions of partitions of 5 and second that we need at most 4 variables in the single monomial whose coefficient we are trying to find, so x_1, x_2, x_3 and x_4 is all we will need (and by this symmetry sometimes even less). Each valid possibility of these variables (discussed below) corresponds to a monomial symmetric function, where its coefficient in the sum will be the possible choices that we can make to find the particular choice of variables appear.

Now:

$$m_{(2,1)} = \sum_{i\neq j} x_i^2 x_j, \quad m_{(1,1)} = \sum_{i<j} x_i x_j,$$

so the possibilities with only four variables are $x_1^3x_2^2$, $x_1^3x_2x_3$, $x_1^2x_2^2x_3$, $x_1^2x_2x_3x_4$ (these are the valid possibilities mentioned above), so we will have $m_{(2,1)}m_{(1,1)}$ as a sum of $m_{(3,2)}$, $m_{(3,1,1)}$, $m_{(2,2,1)}$, $m_{(2,1,1,1)}$. We now see in how many ways can we obtain each, and this will give us the coefficients in which they appear in the multiplication:

- $x_1^3x_2^2$: the x_2^2 cannot be obtained from $m_{(2,1)}$ since the coefficients in $m_{(1,1)}$ are different and thus will never be able to multiply to x_1^2. Thus we must have $x_1^2x_2$ from $m_{(2,1)}$ and hence x_1x_2 from $m_{(1,1)}$. We only have 1 choice.

- $x_1^3x_2x_3$: we need an x_1^2 coming from $m_{(2,1)}$, and then we have $x_1^2x_i$ coming from $m_{(2,1)}$ with $i \in \{2,3\}$ since $i \neq 1$. This means that from $m_{(1,1)}$ we need to have x_1x_j with $j \in \{2,3\}\setminus\{i\}$, which is determined after choosing i. Since we only have the choice of i, we have 2 choices.

- $x_1^2x_2^2x_3$: if x_1^2 is coming from $m_{(2,1)}$, since we cannot have x_2^2 coming from $m_{(1,1)}$ as discussed above, we must have $x_1^2x_2$ coming from $m_{(2,1)}$ and thus x_2x_3 comes from $m_{(1,1)}$; analogously if x_2^2 is coming from $m_{(2,1)}$ we must have x_2x_3 coming from $m_{(1,1)}$. We have the choice of picking whether x_1^2 or x_2^2 comes from $m_{(2,1)}$, so we have 2 choices.

- $x_1^2x_2x_3x_4$: the x_1^2 needs to come from $m_{(2,1)}$, so we have $x_1^2x_i$ coming from $m_{(2,1)}$ with $i \in \{2,3,4\}$ since $i \neq 1$. This means that from $m_{(1,1)}$ we need to have x_jx_k with $j,k \in \{2,3,4\}\setminus\{i\}$, which is determined after choosing i since $j \leq k$. Since we only have the choice of i, we have 3 choices.

Hence the desired result:

$$m_{(2,1)}m_{(1,1)} = m_{(3,2)} + 2m_{(3,1,1)} + 2m_{(2,2,1)} + 3m_{(2,1,1,1)}.$$

Exercise 5.2.3.

Show that if $\mu \leq \lambda$ (in reverse dominance order), then λ precedes μ in the reverse lexicographic order.

As stated, this is false since $\mu = (2,1) \leq (1,1,1) = \lambda$ in the reverse dominance order ($2 > 1$ and $3 > 2$) but $(1,1,1)$ does not precede $(2,1)$ in the reverse lexicographic order since $2 > 1$. We will instead prove: if $\mu \leq \lambda$ (in reverse dominance order), then μ precedes λ in the reverse lexicographic order.

Suppose that $\mu \leq \lambda$ in the reverse dominance order, say $\mu = (\mu_1, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_l)$. Notice that if $l < m$ then $\lambda_1 + \cdots + \lambda_l = n > \mu_1 + \cdots + \mu_l$, a contradiction with $\mu \leq \lambda$ in the reverse dominance order, so $l \geq m$. Since the reverse lexicographic order is a total order, we can compare μ and λ. If they are equal, we are done. If not, we proceed by contradiction; suppose that λ precedes μ in the reverse lexicographic order. This means that there is a $i \in \{1, \ldots, m\}$ such that $\lambda_j = \mu_j$ for $j < i$ and $\lambda_i > \mu_i$ (this can happen since $l \geq m$, so we never run out of space for λ). This implies $\mu_1 + \cdots + \mu_i < \lambda_1 + \cdots + \lambda_i$, a contradiction with $\mu \leq \lambda$ in the reverse dominance order. Hence λ cannot precede μ, and since they are not equal and the reverse lexicographic order is a total order, we must have that μ precedes λ in the lexicographic order, as desired.

Exercise 5.3.2.

Compute the specialized symmetric functions $m_\lambda(x_1,x_2,x_3)$, $e_\lambda(x_1,x_2,x_3)$, $h_\lambda(x_1,x_2,x_3)$, and $p_\lambda(x_1,x_2,x_3)$, for all partitions λ of 3. Compute the specialized symmetric functions $m_\lambda(x_1,x_2)$, $e_\lambda(x_1,x_2)$, $h_\lambda(x_1,x_2)$, and $p_\lambda(x_1,x_2)$, for all partitions λ of 3.

Notice that the three partitions of 3 are (3), $(2,1)$ and $(1,1,1)$. Since we have a lot of polynomials to write, we will omit the intermediate steps that stem from the definition and the further simplifications, and we simply present the final expressions.

We start by $m_\lambda(x_1,x_2,x_3)$:

- $m_{(3)}(x_1,x_2,x_3) = x_1^3 + x_2^3 + x_3^3$,
- $m_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$,
- $m_{(1,1,1)}(x_1,x_2,x_3) = x_1x_2x_3$,

then $e_\lambda(x_1,x_2,x_3)$:

- $e_{(3)}(x_1,x_2,x_3) = x_1x_2x_3$,
- $e_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 3x_1x_2x_3$,
- $e_{(1,1,1)}(x_1,x_2,x_3) = x_1^3+x_2^3+x_3^3+3x_1^2x_2+3x_1x_2^2+3x_1^2x_3+3x_1x_3^2+3x_2^2x_3+3x_2x_3^2+6x_1x_2x_3$,

then $h_\lambda(x_1,x_2,x_3)$:

- $h_{(3)}(x_1,x_2,x_3) = x_1^3+x_2^3+x_3^3+x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2+x_1x_2x_3$,
- $h_{(2,1)}(x_1,x_2,x_3) = x_1^3+x_2^3+x_3^3+2x_1^2x_2+2x_1x_2^2+2x_1^2x_3+2x_1x_3^2+2x_2^2x_3+2x_2x_3^2+3x_1x_2x_3$,
- $h_{(1,1,1)}(x_1,x_2,x_3) = x_1^3+x_2^3+x_3^3+3x_1^2x_2+3x_1x_2^2+3x_1^2x_3+3x_1x_3^2+3x_2^2x_3+3x_2x_3^2+6x_1x_2x_3$,

then $p_\lambda(x_1,x_2,x_3)$:

- $p_{(3)}(x_1,x_2,x_3) = x_1^3 + x_2^3 + x_3^3$,
- $p_{(2,1)}(x_1,x_2,x_3) = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$,
- $p_{(1,1,1)}(x_1,x_2,x_3) = x_1^3+x_2^3+x_3^3+3x_1^2x_2+3x_1x_2^2+3x_1^2x_3+3x_1x_3^2+3x_2^2x_3+3x_2x_3^2+6x_1x_2x_3$.

We now set $x_3 = 0$ in the above and obtain $m_\lambda(x_1,x_2)$:

- $m_{(3)}(x_1,x_2) = x_1^3 + x_2^3$,
- $m_{(2,1)}(x_1,x_2) = x_1^2x_2 + x_1x_2^2$,
- $m_{(1,1,1)}(x_1,x_2) = 0$,

then $e_\lambda(x_1, x_2)$:

- $e_{(3)}(x_1, x_2) = 0$,
- $e_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$,
- $e_{(1,1,1)}(x_1, x_2) = x_1^3 + x_2^3 + 3x_1^2 x_2 + 3x_1 x_2^2$,

then $h_\lambda(x_1, x_2)$:

- $h_{(3)}(x_1, x_2) = x_1^3 + x_2^3 + x_1^2 x_2 + x_1 x_2^2$,
- $h_{(2,1)}(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 x_2 + 2x_1 x_2^2$,
- $h_{(1,1,1)}(x_1, x_2) = x_1^3 + x_2^3 + 3x_1^2 x_2 + 3x_1 x_2^2$,

then $p_\lambda(x_1, x_2)$:

- $p_{(3)}(x_1, x_2) = x_1^3 + x_2^3$,
- $p_{(2,1)}(x_1, x_2) = x_1^3 + x_2^3 + x_1^2 x_2 + x_1 x_2^2$,
- $p_{(1,1,1)}(x_1, x_2) = x_1^3 + x_2^3 + 3x_1^2 x_2 + 3x_1 x_2^2$.

This is what we wanted.

Exercise 5.4.2.

Compute the Schur function $s_{(2,1)}$ in terms of monomial symmetric functions.

We know that for a partition λ of n we have $s_\lambda = \sum_{\mu \geq \lambda} K_{\lambda\mu} m_\mu$. Since $(2,1)$ is a partition of 3, first want to know the partitions of 3 (computed above) that are bigger or equal to this one, which are $(2,1)$ and $(1,1,1)$. Now $K_{\lambda\mu}$ is the number of SSYT of type λ and shape μ. Thus $K_{(2,1)(2,1)} = 1$, we have to put the ones in the top row and the two in the bottom row, and $K_{(2,1)(1,1,1)} = 2$ since given a SYT of shape $(2,1)$ we need the one in the top left box, we have two places where we can put the 2 and then the 3 has to fill the remaining space, so we only have 2 choices.

Hence the desired result:

$$s_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}.$$

References

[1] A. Prasad, *Representation Theory: A Combinatorial Viewpoint*, Cambridge studies in advanced mathematics, 2015.

[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.