Representations of Finite Groups - Homework 9

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Exercise 5.4.13.

Show that:

$$\operatorname{trace}(w_{(n)}; V_\lambda) = \begin{cases} (-1)^k \text{ if } \lambda = (n-k, 1^k) \text{ for some } 0 \leq k \leq n-1, \\ 0 \text{ otherwise.} \end{cases}$$

In virtue of [1, Theorem 5.4.11. (p. 115)] we have that $\operatorname{trace}(w_(n); V_\lambda)$ is the coefficient of $x^{\lambda+\delta}$ in $p_{(n)}a_\delta$. Notice that since (n) is a partition of n, we can specialize to n variables, thus taking $\delta = (n-1, n-2, \dots, 1, 0)$. Hence we have to find the coefficient of $x^{\lambda+\delta}$ in:

$$p_{(n)}(x_1, \dots, x_n)a_\delta = \left(\sum_{i=1}^{n} x_i^n\right)\left(\prod_{1\leq i<j\leq n} x_i - x_j\right).$$

Since $p_{(n)}(x_1, \dots, x_n)a_\delta$ is a multiplication of a symmetric polynomial with an alternating polynomial, it is an alternating polynomial, and since we saw that alternating polynomials are determined by the coefficients of their decreasing monomials, it is enough to determine the coefficient of one of these. If any of these decreasing monomials can be related to $x^{\lambda+\delta}$, we would have its coefficient and thus as above the desired result. Hence we now proceed to find a relationship between the decreasing monomials and $x^{\lambda+\delta}$.

First, note that by definition a_δ is the Vandermonde determinant:

$$a_\delta = \begin{vmatrix} x_1^{n-1} & \cdots & x_1^0 \\ \vdots & & \vdots \\ x_n^{n-1} & \cdots & x_n^0 \end{vmatrix} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{n-i}$$

by the Leibniz formula for determinants. Thus we can rewrite:

$$p_{(n)}(x_1, \dots, x_n)a_\delta = \left(\sum_{i=1}^{n} x_i^n\right)\left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{n-i}\right)$$

so each monomial in $p_{(n)}(x_1, \dots, x_n)a_\delta$ is of the form:

$$\operatorname{sgn}(\sigma) x_i^n x_{\sigma(1)}^{n-1} \cdots x_{\sigma(n-1)}^1.$$

Since we are looking for decreasing monomials we need $i = 1$ because $n > n-k$ for all $1 \leq k \leq n-1$. Moreover since $\sigma \in S_n$ there is an $j \in \{1, \dots, n\}$ such that $\sigma(j) = 1$. Hence we can rewrite:

$$\operatorname{sgn}(\sigma) x_1^n x_{\sigma(1)}^{n-1} \cdots x_{\sigma(n-1)}^1 = \operatorname{sgn}(\sigma) x_1^{n+n-j} x_{\sigma(1)}^{n-1} \cdots x_{\sigma(j-1)}^{n-(j-1)} x_{\sigma(j+1)}^{n-(j+1)} \cdots x_{\sigma(n-1)}^1.$$

Since we are looking for decreasing monomials and the monomial above already has decreasing exponents, we just need:

$$\sigma(1) < \cdots < \sigma(j-1) < \sigma(j+1) < \cdots < \sigma(n-1)$$

and since $\sigma(j) = 1$ we have that none of the above can take 1 as a value, hence:

$$1 < \sigma(1) < \cdots < \sigma(j-1) < \sigma(j+1) < \cdots < \sigma(n-1) \leq n.$$

Notice that since we are looking for decreasing monomials, we can choose σ such that:

$$\sigma(1) = 2, \sigma(2) = 3, \ldots, \sigma(j-1) = j, \sigma(j+1) = j+1, \ldots, \sigma(n-1) = n-1$$

and the monomial will be decreasing. There are other possible options for σ, but the one chosen here is the simpler one (and, as we will see, the one that yields the monomial $x^{\lambda+\delta}$, so this is the correct choice for our intentions). Hence this σ is the cycle $(1, 2, \ldots, j-1, j) = (1,2)(2,3)\cdots(j-2, j-1)(j-1, j)$ having a decomposition into $j-1$ transpositions and thus $\mathrm{sgn}(\sigma) = (-1)^{j-1}$. We then have for our choice of σ:

$$\mathrm{sgn}(\sigma) x_1^n x_{\sigma(1)}^{n-1} \cdots x_{\sigma(n-1)}^1 = (-1)^{j-1} x_1^{n+n-j} x_2^{n-1} \cdots x_{j-1}^{n-(j-2)} x_j^{n-(j-1)} x_{j+1}^{n-(j+1)} \cdots x_{n-1}^1$$

which has as exponents the tuple:

$$\begin{aligned}
&(n{+}n-j, n-1, \ldots, n-j+1, n-j-1, \ldots, 1, 0) = \\
&\quad = (n-(j-1) + (n-1), 1 + (n-2), \ldots, 1 + (n-j), 1 + (n-(j+1)), \ldots, 1, 0) \\
&\quad = (n-(j-1), 1, \ldots, 1) + (n-1, n-2, \ldots, 1, 0)
\end{aligned}$$

so setting $k = j-1$ we have $k \in \{0, \ldots, n-1\}$. Moreover we have that if $\lambda = (n-k, 1, \ldots, 1)$ then our decreasing monomial is:

$$(-1)^{j-1} x_1^{n+n-j} x_2^{n-1} \cdots x_{j-1}^{n-(j-2)} x_j^{n-(j-1)} x_{j+1}^{n-(j+1)} \cdots x_{n-1}^1 = (-1)^k x^{\lambda+\delta}$$

moreover, notice that since the tuple δ has a zero in the position n, we need to have 0 in the exponent of x_n to be able to decompose the exponents of the decreasing monomial as a sum of a partition λ with δ. In particular this means that if we make a choice of σ such that x_n appears in the decreasing monomial, it will not contribute to the coefficient of $x^{\lambda+\delta}$, regardless of the partition λ. Hence the condition of having $\lambda = (n-k, 1\ldots, 1)$ is not only sufficient but necessary for $x^{\lambda+\delta}$ to have non zero coefficient.

We thus found that the monomial $x^{\lambda+\delta}$ appears in $p_{(n)}(x_1, \ldots, x_n) a_\delta$ if and only if $\lambda = (n-k, 1\ldots, 1)$ for some $0 \leq k \leq n-1$, and in that case it appears with coefficient $(-1)^k$. Hence:

$$\mathrm{trace}(w_{(n)}; V_\lambda) = \begin{cases} (-1)^k \text{ if } \lambda = (n-k, 1^k) \text{ for some } 0 \leq k \leq n-1, \\ 0 \text{ otherwise} \end{cases}$$

as desired.

Exercise 5.5.2.

Show that $\mathrm{ch}_n(\chi_\lambda) = s_\lambda$ for every partition λ of n.

Notice that:

$$\langle \mathrm{ch}_n(\chi_\lambda), s_\mu \rangle = \langle \chi_\lambda, \chi_\mu \rangle_{S_n} = \begin{cases} 1 \text{ if } \lambda = \mu, \\ 0 \text{ if } \lambda \neq \mu, \end{cases}$$

in virtue of [1, Theorem 5.5.1. (p. 118)] and Schur's orthogonality relations. Since the Schur functions $\{s_\lambda : \lambda \vdash n\}$ form an orthonormal basis on Λ_K^n, this yields $\mathrm{ch}_n(\chi_\lambda) = s_\lambda$ as desired.

Exercise 5.5.3.

Show that for all class functions $f, g \in K[S_n]$ we have $\langle \mathrm{ch}_n(f), \mathrm{ch}_n(g)\rangle = \langle f, g\rangle_{S_n}$.

We saw in [1, Exercise 1.7.12. (p. 26)] that the irreducible characters form a basis for the class functions. Set this basis as $\{\chi_{\lambda_1}, \dots, \chi_{\lambda_m}\}$ and write $f = \sum_{i=1}^m f_{\chi_{\lambda_i}} \chi_{\lambda_i}$, $g = \sum_{i=1}^m g_{\chi_{\lambda_i}} \chi_{\lambda_i}$. Thus:

$$\begin{aligned}
\langle \mathrm{ch}_n(f), \mathrm{ch}_n(g)\rangle &= \langle \mathrm{ch}_n(f), \sum_{i=1}^m g_{\chi_{\lambda_i}} \mathrm{ch}_n(\chi_{\lambda_i})\rangle = \sum_{i=1}^m g_{\chi_{\lambda_i}} \langle \mathrm{ch}_n(f), \mathrm{ch}_n(\chi_{\lambda_i})\rangle \\
&= \sum_{i=1}^m g_{\chi_{\lambda_i}} \langle \mathrm{ch}_n(f), s_{\lambda_i}\rangle = \sum_{i=1}^m g_{\chi_{\lambda_i}} \langle f, \chi_{\lambda_i}\rangle_{S_n} = \langle f, \sum_{i=1}^m g_{\chi_{\lambda_i}} \chi_{\lambda_i}\rangle_{S_n} \\
&= \langle f, g\rangle_{S_n}
\end{aligned}$$

where we have used the bilinearity of the form on Λ_K^n and the bilinearity of the map on $K[S_n]$, as well as [1, Theorem 5.5.1. (p. 118)] and [1, Exercise 5.5.2. (p. 118)].

Exercise 5.5.5.

Here we use the abbreviated notation mentioned on [1, (p. 118)], namely $\mathrm{ch}_n(V_\lambda) = s_\lambda$, and also the usual abuse of notation by identifying the vector spaces of a representation with the morphism of that representation. For every partition λ of n, we show that:

1. We have:

$$\begin{aligned}\mathrm{ch}_n(K[X_\lambda]) = \mathrm{ch}_n\left(\bigoplus_{\nu\leq\lambda} V_\nu^{\oplus K_{\nu\lambda}}\right) &= \sum_{\nu\leq\lambda}\mathrm{ch}_n(V_\nu^{\oplus K_{\nu\lambda}}) \\ &= \sum_{\nu\leq\lambda} K_{\nu\lambda}\mathrm{ch}_n(V_\nu) = \sum_{\nu\leq\lambda} K_{\nu\lambda}s_\nu = h_\lambda\end{aligned}$$

   where we have used [1, Theorem 3.3.1. (p. 68)], linearity of the map ch_n and [1, Equation 5.19 (p. 110)]. Thus $\mathrm{ch}_n(K[X_\lambda]) = h_\lambda$.

2. We have:

$$\begin{aligned}\mathrm{ch}_n(K[X_\lambda]\otimes\epsilon) = \mathrm{ch}_n\left(\bigoplus_{\nu'\leq\lambda} V_\nu^{\oplus K_{\nu'\lambda}}\right) &= \sum_{\nu'\leq\lambda}\mathrm{ch}_n(V_\nu^{\oplus K_{\nu'\lambda}}) \\ &= \sum_{\nu'\leq\lambda} K_{\nu'\lambda}\mathrm{ch}_n(V_\nu) = \sum_{\nu\leq\lambda} K_{\nu\lambda}\mathrm{ch}_n(V_{\nu'}) = \sum_{\nu\leq\lambda} K_{\nu\lambda}s_{\nu'} = e_\lambda\end{aligned}$$

   where we have used the direct consequence of [1, Theorem 4.4.2. (p. 68)] that $K[X_\lambda]\otimes\epsilon = \bigoplus_{\nu'\leq\lambda} V_\nu^{\oplus K_{\nu'\lambda}}$, linearity of the map ch_n and [1, Equation 5.18 (p. 110)]. Thus $\mathrm{ch}_n(K[X_\lambda]\otimes\epsilon) = e_\lambda$.

Exercise 5.6.4.

Let f_λ denote the number of Standard Young Tableaux of shape λ. We show that for every partition λ we have $f_\lambda = \sum_{\mu\in\lambda^-} f_\mu$, and as a consequence we recover that $\dim(V_\lambda) = f_\lambda$.

Given any partition ν of n we define SYT_ν the set of Standard Young Tableaux of shape ν. To prove that $f_\lambda = \sum_{\mu\in\lambda^-} f_\mu$ it is enough to define a bijection between SYT_λ and $\coprod_{\mu\in\lambda^-} SYT_\mu$. We now construct this bijection.

Given a SYT of shape λ (that is, an element in SYT_λ), we have that the Young diagram is filled with the numbers $\{1,\dots,n\}$, each appearing only once. We can then remove the box labeled n, obtaining a SYT of shape $\mu \in \lambda^-$ (that is, an element in $\coprod_{\mu\in\lambda^-} SYT_\mu$).

Given a SYT of shape $\mu \in \lambda^-$, its Young diagram has $n-1$ boxes and its shape was obtained from the Young diagram of λ by removing one box. This means that there is a unique way of adding a box to μ and obtain a Young diagram of shape λ, and filling this box with the label n we would obtain a SYT of shape λ.

The two maps above are well defined and inverses of each other, both by construction. Hence they establish the desired bijection between SYT_λ and $\coprod_{\mu\in\lambda^-} SYT_\mu$, proving that $f_\lambda = \sum_{\mu\in\lambda^-} f_\mu$.

To prove that $\dim(V_\lambda) = f_\lambda$ we proceed by induction on n the number that λ partitions. The case $n = 1$ follows directly from [1, Exercise 3.3.4. (p. 69)]: that gives us that $f_\lambda = 1$ and since $K[S_1]$ is one dimensional we have that V_λ is one dimensional. Suppose now that $\dim(V_\nu) = f_\nu$ holds for ν any partition of n, for any partition λ of $n+1$ we use [1, Theorem 5.6.2. (p. 120)] and what we just proved: $\dim(V_\lambda) = \sum_{\mu\in\lambda^-} \dim(V_\mu) = \sum_{\mu\in\lambda^-} f_\mu = f_\lambda$, as desired. This finishes induction and thus indeed $\dim(V_\lambda) = f_\lambda$ for λ any partition of any $n \in \mathbb{N}$.

References

[1] A. Prasad, *Representation Theory: A Combinatorial Viewpoint*, Cambridge studies in advanced mathematics, 2015.

[2] T. W. Hungerford, *Algebra*, Springer-Verlag, 1974.