Topological Quantum Computation - Homework

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Exercise 1

Unfortunately I could not learn in time how to use TikZ properly to type this exercise properly. Because of this, here I present my results but not necessarily how to obtain them.

1. Let $\mathcal{L} = \{0, 1\}$ with $\hat{1} = 1$. Suppose that N_{ij}^k is fully symmetric and $N_{11}^1 = 1$. Compute the dimension of the Hilbert space obtained by setting N anyons labeled 1 in a disk with boundary labeled 0.

   We will name $\mathcal{H}_{N,x}^y$ the Hilbert space formed by setting $N-1$ anyons labeled 1 and one anyon labeled $x \in \mathcal{L}$ (for a total of N anyons) in a disk with boundary labeled y (writing $\mathcal{H}_{N,x} = \mathcal{H}_{N,x}^0$). Applying cuts and using the properties to manipulate them, we obtain a recursion formula:

   (i) For $N = 0$ we have that $\mathcal{H}_{0,\cdot}$ is a surface of genus 0, so we will treat this in the next part of the exercise.

   (ii) For $N = 1$ we have that $\mathcal{H}_{1,1}$ is an empty cylinder with one end labeled 1 and the other end labeled 0, thus has dimension 0 (this is one of the axioms).

   (iii) For $N = 2$ we have that $\mathcal{H}_{2,1}$ is a pants figure with labels 1, 1 and 0 (the order does not matter since N_{ij}^k is fully symmetric) so it has dimension $N_{11}^0 = 1$ (applying one of the axioms).

   (iv) For $N \geq 3$, we have that $\mathcal{H}_{N+1,1}$ can be separated into pants with bottom labeled 0 and tops labeled 1. This can be cut along the last pants, so using the axioms we obtain:

   $$\mathcal{H}_{N+1,1} = \bigoplus_{x \in \mathcal{L}} \mathcal{H}_{N,x} \otimes \mathcal{H}_{2,1}^x = \mathcal{H}_{N,1} \otimes \mathbb{C} \oplus \mathcal{H}_{N,0} \otimes \mathbb{C} \cong \mathcal{H}_{N,1} \oplus \mathcal{H}_{N,0}$$

   where in the second equality we used that $N_{11}^1 = 1$ and $N_{11}^0 = 1$, and in the third the fact that we are taking tensor products over $\mathbb{C}$. Since anything labeled 0 may be removed (it is one of the axioms), the above is:

   $$\mathcal{H}_{N+1,1} \cong \mathcal{H}_{N,1} \oplus \mathcal{H}_{N-1,1}$$

   meaning that:

   $$\dim(\mathcal{H}_{N+1,1}) = \dim(\mathcal{H}_{N,1}) + \dim(\mathcal{H}_{N-1,1})$$

   the Fibonacci recursion. Since we saw that $\dim(\mathcal{H}_{1,1}) = 1$ and $\dim(\mathcal{H}_{2,1}) = 1$, this completely determines the desired dimension.

2. Let $\mathcal{L} = \{0, 1, 2\}$ with $\hat{1} = 1$ and $\hat{2} = 2$. Suppose that N_{ij}^k is fully symmetric, $N_{22}^1 = 1$, $N_{11}^2 = 0$, $N_{22}^2 = 0$, $N_{11}^1 = 0$. Compute the dimension of a genus N surface.

We now establish some notation (unfortunately, different from the section above): name ${}^x\mathcal{H}_N^y$ the Hilbert space formed by a surface of genus N with an anyon on the left labeled x and an anyon on the right labeled y. If $x = 0$ or $y = 0$, we may omit the label and write $\mathcal{H}_N^y$ or ${}^x\mathcal{H}_N$ or $\mathcal{H}_N$.

We now write some of the dimensions of these spaces that can be computed directly by the axioms (or that not more than a single cut is needed):

$$\begin{aligned} &\dim(\mathcal{H}_1) = 3, \quad \dim({}^1\mathcal{H}_1) = 1, \quad \dim({}^2\mathcal{H}_1) = 1, \\ &\dim(\mathcal{H}_1^1) = 1, \quad \dim({}^1\mathcal{H}_1^1) = 3, \quad \dim({}^2\mathcal{H}_1^1) = 1, \\ &\dim(\mathcal{H}_1^2) = 0, \quad \dim({}^1\mathcal{H}_1^2) = 0, \quad \dim({}^2\mathcal{H}_1^2) = 3 \end{aligned}$$

By the same procedure as above, separating the genus N surface into a surface of genus $N-1$ and another of genus 1 by cutting (and thus labeling the cut with elements in $\mathcal{L}$). This yields:

$$\mathcal{H}_N = \bigoplus_{x \in L} \mathcal{H}_{N-1}^x \otimes^x \mathcal{H}_1 = \mathcal{H}_{N-1} \otimes \mathcal{H}_1 \oplus \mathcal{H}_{N-1}^1 \otimes^1 \mathcal{H}_1 \oplus \mathcal{H}_{N-1}^2 \otimes^2 \mathcal{H}_1$$

and thus when we compute the dimensions:

$$\dim(\mathcal{H}_N) = 3\dim(\mathcal{H}_{N-1}) + \dim(\mathcal{H}_{N-1}^1).$$

Similarly:

$$\mathcal{H}_N^1 = \bigoplus_{x \in L} \mathcal{H}_{N-1}^x \otimes^x \mathcal{H}_1^1 = \mathcal{H}_{N-1} \otimes \mathcal{H}_1^1 \oplus \mathcal{H}_{N-1}^1 \otimes^1 \mathcal{H}_1^1 \oplus \mathcal{H}_{N-1}^2 \otimes^2 \mathcal{H}_1^1$$

and thus when we compute the dimensions:

$$\dim(\mathcal{H}_N^1) = \dim(\mathcal{H}_{N-1}) + 3\dim(\mathcal{H}_{N-1}^1) + \dim(\mathcal{H}_{N-1}^2).$$

And finally:

$$\mathcal{H}_N^2 = \bigoplus_{x \in L} \mathcal{H}_{N-1}^x \otimes^x \mathcal{H}_1^2 = \mathcal{H}_{N-1} \otimes \mathcal{H}_1^2 \oplus \mathcal{H}_{N-1}^2 \otimes^1 \mathcal{H}_1^2 \oplus \mathcal{H}_{N-1}^2 \otimes^2 \mathcal{H}_1^2$$

and thus when we compute the dimensions:

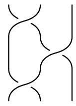
$$\dim(\mathcal{H}_N^2) = 3\dim(\mathcal{H}_{N-1}^2).$$

We now have three iterations and the respective initial values that enable us to compute $\dim(\mathcal{H}_N)$ for every N, the desired result.

Exercise 2

1. Check that $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ using the "bowling alley" idea.

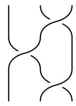
   The braid $\sigma_1\sigma_2\sigma_1$ can be drawn as:

   and thus setting $1-t$ the probability of falling and t the probability of continuing straight, the matrix governing this bowling alley is:

   $$\rho(\sigma_1\sigma_2\sigma_1) = \begin{bmatrix} (1-t)(1-t)+t(1-t) & (1-t)t & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   The braid $\sigma_2\sigma_1\sigma_2$ can be drawn as:

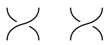
   and the matrix governing this bowling alley is:

   $$\rho(\sigma_2\sigma_1\sigma_2) = \begin{bmatrix} 1-t & t(1-t) & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

   and since we indeed have $(1-t)(1-t)+t(1-t) = 1+t^2-2t+t-t^2 = 1-t$ this yields $\rho(\sigma_1\sigma_2\sigma_1) = \rho(\sigma_2\sigma_1\sigma_2)$ which is what we wanted to prove.

2. Check that $\sigma_1\sigma_3 = \sigma_3\sigma_1$ using the "bowling alley" idea.

   I could not manage to draw the difference between $\sigma_1\sigma_3$ and $\sigma_3\sigma_1$. I now describe them stating from the following diagram:

   to obtain $\sigma_1\sigma_3$ we need to pull the second crossing up and extend the first pair of strings accordingly, and to obtain $\sigma_3\sigma_1$ we need to pull the first crossing up and extend the second pair of strings accordingly.

   With these pictures in mind, the matrix governing the bowling alley of $\sigma_1\sigma_3$ is:

   $$\rho(\sigma_1\sigma_3) = \begin{bmatrix} t & 1-t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & t & 1-t \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

and the matrix governing the bowling alley of $\sigma_3\sigma_1$ is:

$$\rho(\sigma_3\sigma_1) = \begin{bmatrix} t & 1-t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & t & 1-t \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

and we indeed have $\rho(\sigma_1\sigma_3) = \rho(\sigma_3\sigma_1)$ which is what we wanted to prove.

Exercise 3

We check that for $t \in C$ (possibly treated as a variable) the vectors:

$$v_1 = \begin{bmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad, \quad v_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -t \\ 1 \end{bmatrix}$$

span a space W that is $\tilde{\rho}(B_n)$ invariant, that $W \cap \mathbb{C}v_0 = \{0\}$ and that $W \oplus \mathbb{C}v_0 = \mathbb{C}^n$ (recall that v_0 is the sum of the elements of the canonical basis). First, notice that these v_i for $i = 1, \ldots, n-1$ form a basis of W, since they are linearly independent (they have zero and non zero entries in places where the others cannot reach them).

To prove the above, it is enough to do so for elements of the basis of B_n. Hence let $\sigma_i \in B_n$ for $i = 1, \ldots, n-1$. Then:

$$\tilde{\rho}(\sigma_i) = \begin{bmatrix} 1_{(i-1)\times(i-1)} & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & 1_{(n-i-1)\times(n-i-1)} \end{bmatrix}$$

and thus we have to divide the multiplication $\tilde{\rho}(\sigma_i)v_j$ into a few cases:

1. $j < i-1$ or $j \geq i+2$ we have $\tilde{\rho}(\sigma_i)v_j = v_j$ by multiplying using then bloc matrices.
2. $j = i-1$ we have:

$$\tilde{\rho}(\sigma_i)v_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -t \\ 1-t \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v_j + v_{j+1}$$

since the first non zero entry is in the position j.

3. $j = i$ we have:

$$\tilde{\rho}(\sigma_i)v_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -t(1-t)+t \\ -t \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ t^2 \\ -t \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -tv_j$$

since the first non zero entry is in the position j.

4. $j = i + 1$ we have:

$$\tilde{\rho}(\sigma_i)v_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ t^2 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = tv_{j-1} + v_j$$

since the first non zero entry is in the position $j - 1$.

Since $t \in \mathbb{C}$, we can always rewrite this as an element in W and thus $\tilde{\rho}(\sigma_i)(W) \subseteq W$ for all $i = 1, \dots, n-1$, so $\tilde{\rho}(B_n)(W) \subseteq W$ and hence W is invariant.

Suppose now that $W \cap \mathbb{C}v_0 \neq \{0\}$, that is, $v_0 \in W$. This means that there exist $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$ such that $v_0 = \alpha_1 v_1 + \cdots + \alpha_{n-1}v_{n-1}$. Now, notice that since v_j for $2 \leq j \leq n-1$ has a 0 in the first component, we need $\alpha_1 = -1/t$, and thus:

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = v_0 = \begin{bmatrix} 1 \\ \frac{-1}{t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_2 v_2 + \cdots + \alpha_{n-1}v_{n-1}$$

As above, v_j for $3 \leq j \leq n-1$ has a 0 in the second component, we need $-1/t - t\alpha_2 = 1$ so $\alpha_2 = -1/t - 1/t^2$. A straightforward induction yields that $\alpha_i = -1/t - \cdots - 1/t^i$ for $i = 1, \dots, n-1$. This means that the equality $v_0 = \alpha_1 v_1 + \cdots + \alpha_{n-1}v_{n-1}$ in the last component yields:

$$1 = \alpha_{n-1} \iff 1 = -\frac{1}{t} - \cdots - \frac{1}{t^{n-1}} \iff t^{n-1} + \cdots + t + 1 = 0.$$

We then need to distinguish our treatment of $t \in \mathbb{C}$. If t is simply a complex variable, then the above polynomial has no solutions and we have a contradiction with $W \cap \mathbb{C}v_0 \neq \{0\}$. If t is a complex number not satisfying the above polynomial, we also have a contradiction with $W \cap \mathbb{C}v_0 \neq \{0\}$. If t is a complex number solution of the polynomial, then $W \cap \mathbb{C}v_0 = \mathbb{C}v_0$.

Now in the cases where t is not a solution of the polynomial $t^{n-1} + \cdots + t + 1 = 0$, we check that $W \oplus \mathbb{C}v_0 = \mathbb{C}^n$. For this we check the linear independence of $v_0, \ldots, v_{n-1}$: suppose that there are $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{C}$ such that:

$$\sum_{i=0}^{n-1} \alpha_i v_i = 0 \Longrightarrow \alpha_0 v_0 = -\alpha_1 v_1 - \cdots - \alpha_{n-1} v_{n-1} \Longrightarrow v_0 = -\frac{\alpha_1}{\alpha_0} v_1 - \cdots - \frac{\alpha_{n-1}}{\alpha_0} v_{n-1}$$

which would mean that $v_0 \in W \cap \mathbb{C}v_0 = \{0\}$, a contradiction. Hence no such $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ exist, so $W + \mathbb{C}v_0 = \mathbb{C}^n$. Since $W \cap \mathbb{C}v_0 = \{0\}$ this yields $W \oplus \mathbb{C}v_0 = \mathbb{C}^n$.

Exercise 4

Let $\tilde{\rho}$ be the Bureau representation and ρ the reduced Bureau representation. We prove that for any $\beta \in B_n$ we have:

$$\frac{\det(M(\beta))}{1+t+\cdots+t^{n-1}} = \det(\tilde{M}(\beta)(1,1)).$$

First notice that the matrix:

$$P = \begin{bmatrix} v_0 \cdots v_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -t & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & -t \\ 1 & 0 & \cdots & 1 \end{bmatrix}$$

is the change of basis matrix from the canonical basis $e_1, \dots, e_n$ to the basis $v_0, \dots, v_{n-1}$. Notice that by induction $\det(P) = t^{n-1} + t^{n-2} + \cdots + t + 1$. Since v_0 corresponds to the trivial representation in the decomposition of $\tilde{\rho}$ (because it is the vector with all ones), this means that:

$$\tilde{\rho}(\beta) = P^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \rho(\beta) \end{bmatrix} P$$

and thus:

$$\begin{aligned} \tilde{M}(\beta) = 1_n - \tilde{\rho}(\beta) = 1_n - P^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \rho(\beta) \end{bmatrix} P = P^{-1}P - P^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \rho(\beta) \end{bmatrix} P \\ = P^{-1} \left(1_n - \begin{bmatrix} 1 & 0 \\ 0 & \rho(\beta) \end{bmatrix} \right) P = P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1_{n-1} - \rho(\beta) \end{bmatrix} P = P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & M(\beta) \end{bmatrix} P \end{aligned}$$

and since $P^{-1} = \operatorname{adj}(P)/\det(P)$, we now have a path towards computing this. Notice that multiplying:

$$P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & M(\beta) \end{bmatrix} P = P^{-1}(\bullet, 1) M(\beta) P(1, \bullet)$$

where the notation is analogous to the one used for minors, that is $P^{-1}(\bullet, 1)$ means that we remove the first column and $P(1, \bullet)$ means that we remove the first row. This equality holds because the column of 0 effectively deletes the first row of P, and the row of 0 effectively deletes the first column of P^{-1}. Moreover, when we compute $\tilde{M}(\beta)(1,1)$ we are removing the first row and columns:

$$\tilde{M}(\beta)(1,1) = (P^{-1}(\bullet, 1) M(\beta) P(1, \bullet))(1,1) = P^{-1}(1,1) M(\beta) P(1,1)$$

since because of how matrix multiplication works, removing the first column after multiplying can be interpreted as removing the first column of P before multiplying, and removing the first row can be interpreted as that we removed the first column of P^{-1} before multiplying (where here "can be interpreted" means that the result of both operations are indeed the same). Remark that we have decreased the dimension of each row and column by 1, since P started as an $n \times n$ matrix, we now have all $n-1 \times n-1$ matrices. Hence

$$\begin{aligned}\det(\tilde{M}(\beta)(1,1)) &= \det(P^{-1}(1,1)M(\beta)P(1,1)) = \det(P(1,1)P^{-1}(1,1)M(\beta)) \\ &= \det(P(1,1)P^{-1}(1,1))\det(M(\beta))\end{aligned}$$

so we just have to compute $\det(P(1,1)P^{-1}(1,1))$. Part of this is very easy since:

$$P(1,1) = \begin{bmatrix} 1 & -t & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -t \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The other part of the computation is harder, using that $P^{-1} = \operatorname{adj}(P)/\det(P)$, we compute by induction:

$$\operatorname{adj}(P)(1,1) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^{n-2} \\ -t^{n-2} - \cdots - t & t+1 & t^2+t & \cdots & t^{n-2}+t^{n-3} \\ -t^{n-3} - \cdots - t & -t^{n-2} - \cdots - t^2 & t^2+t+1 & \cdots & t^{n-2}+t^{n-3}+t^{n-4} \\ \vdots & \vdots & \vdots & & \vdots \\ -t & -t^2 & -t^3 & \cdots & t^{n-2}+\cdots+t+1 \end{bmatrix}$$

where the pattern is convoluted but clear; start with the row vector:

$$\begin{bmatrix} 1 \\ -t^{n-2} - t^{n-3} - \cdots - t \\ -t^{n-3} - \cdots - t \\ \vdots \\ -t \end{bmatrix}$$

having 1 in the first component, $t^{n-1}+1-\det(P)$ in the second component, $t^{n-1}+t^{n-2}+1-\det(P)$ in the third component, and so on, in general $t^{n-1}+t^{n-2}+\cdots+t^{n-k+1}+1-\det(P)$ in the k-th component for $k>1$. Notice that this vector has the first entry "positive" and the other entries "negative". Setting this vector as the first column, we obtain the next by first multiplying the positive entry (entries in the following cases) by t, appending a positive entry right below the last one by adding 1 to this last entry, and then multiplying the remaining negative entries by t. The process above removes a negative entry and adds a positive one, so the j-th column should have j positive entries and $n-1-j$ negative entries. The reason why we have noted that there is a way to express the entries in terms of $\det(P)$ is because this is useful when computing the induction. If we multiply, again by induction we obtain:

$$P(1,1)P^{-1}(1,1) = \frac{1}{\det(P)} \begin{bmatrix} t^{n-1}+t^{n-2}+\cdots+t^2+1 & \cdots & -t^{n-1} \\ -t & \cdots & -t^{n-1} \\ \vdots & & \vdots \\ -t & \cdots & t^{n-2}+t^{n-3}+\cdots+t+1 \end{bmatrix}$$

which can be better put as:

$$P(1,1)P^{-1}(1,1) = \frac{1}{\det(P)} \begin{bmatrix} \det(P)-t & \cdots & -t^{n-1} \\ -t & \cdots & -t^{n-1} \\ \vdots & & \vdots \\ -t & \cdots & \det(P)-t^{n-1} \end{bmatrix}$$

$$= \frac{1}{\det(P)} \left(\det(P) 1_{n-1} + \begin{bmatrix} -t & \cdot & -t^{n-1} \\ \vdots & & \vdots \\ -t & \cdots & -t^{n-1} \end{bmatrix} \right)$$

where again this rewriting is useful to prove by induction that:

$$\begin{vmatrix} \det(P)-t & \cdots & -t^{n-1} \\ -t & \cdots & -t^{n-1} \\ \vdots & & \vdots \\ -t & \cdots & \det(P)-t^{n-1} \end{vmatrix} = \det(P)^{n-2},$$

so since we are working with $n-1 \times n-1$ matrices we have:

$$\det(P(1,1)P^{-1}(1,1)) = \frac{1}{\det(P)^{n-1}} \det(P)^{n-2} = \frac{1}{\det(P)}.$$

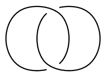
Putting all the above together, we now have:

$$\det(\tilde{M}(\beta)(1,1)) = \det(P(1,1)P^{-1}(1,1)) \det(M(\beta))$$
$$= \frac{\det(M(\beta))}{\det(P)} = \frac{\det(M(\beta))}{t^{n-1} + \cdots + t + 1}$$

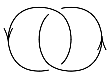
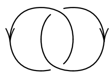
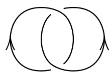
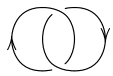
the desired result.

Exercise 5

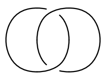
We compute the Jones polynomial for the Hopf link, considering the mirror images and all possible orientations. Notice that each Hopf link has two strands, each strand two possible orientations, so a priori we have a total of eight oriented Hopf links. We name this as follow: consider the Hopf link:



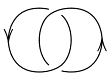
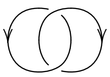
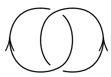
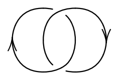
we have the possible orientations of the top pointing right, pointing outwards, pointing inwards or pointing left, as expressed in that order in the following diagrams:

a)  b)  c)  d) 

and if we consider now the mirror image of the Hopf link above:



we again have the same possible orientations, expressed in that order in the following diagrams:

e)  f)  g)  h) 

so we now have names for all the possibilities. Notice that by easy rotations we can tell that $a) \cong d)$, $b) \cong c)$, $e) \cong h$, $f) \cong g)$. Moreover, if we rotate $e)$ by $\pi/2$ so that both crossings are at the same height, and then move the right crossing under the left crossing, we obtain $b)$. Similarly, if we rotate $f)$ by $\pi/2$ so that both crossings are at the same height, and then move the right crossing under the left crossing, we obtain $a)$. This means that we only need to worry about the configurations of $f)$ and $b)$. The reason why we choose these configurations is that they are already oriented in the same way that we orient the braids: they flow from bottom to top, so we can apply our definition of the Jones polynomial for links and it will work out.

First for $L = f)$, notice how it is in fact the closure of σ_1^2 with $e(\sigma_1^2) = 2$, and thus:

$$\begin{aligned}\mathrm{Tr}(\sigma_1^2) &= \mathrm{Tr}((A^{-1}u_1 + A1)^2) = \mathrm{Tr}(A^{-2}u_1^2 + A^2 1 + 2u_1) \\ &= \mathrm{Tr}(A^{-2}du_1 + A^2 1 + 2u_1) = A^{-2}d^2 + A^2 d^2 + 2d\end{aligned}$$

since $\mathrm{Tr}(u_1) = d$ and $\mathrm{Tr}(1) = d^2$. Hence:

$$\begin{aligned}J(L, q) &= \frac{(-A^{-3})^2 \mathrm{Tr}(\sigma_1^2)}{d} = \frac{(-A^{-3})^2(A^{-2}d^2 + A^2 d^2 + 2d)}{d} \\ &= -A^{-10} - A^{-2} = -q^{-5/2} - q^{-1/2}\end{aligned}$$

where we have used that $d = -(A^2 + A^{-2})$ and that $A^4 = q$.

Second for $L = b$), notice how it is in fact the closure of σ_1^{-2} with $e(\sigma_1^2) = -2$, and thus:

$$\begin{aligned} \mathrm{Tr}(\sigma_1^2) &= \mathrm{Tr}((Au_1 + A^{-1}1)^2) = \mathrm{Tr}(A^2 u_1^2 + A^{-2}1 + 2u_1) \\ &= \mathrm{Tr}(A^2 d u_1 + A^{-2}1 + 2u_1) = A^2 d^2 + A^{-2} d^2 + 2d \end{aligned}$$

computing as before. Hence:

$$\begin{aligned} J(L,q) &= \frac{(-A^{-3})^{-2}\mathrm{Tr}(\sigma_1^{-2})}{d} = \frac{(-A^{-3})^{-2}(A^2 d^2 + A^{-2} d^2 + 2d)}{d} \\ &= -A^{10} - A^2 = -q^{5/2} - q^{1/2} \end{aligned}$$

again computing as before. These are the desired results.

Exercise 9

We check that the $CNOT$ matrix is entangling. For this, we will use the equivalent definition saying that a matrix is entangling if and only if it can send a pure state to an entangled state.

We use the standard notation:

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

arising from tensoring the canonical basis $e_1 = |0\rangle$, $e_2 = |1\rangle$ with itself by appending. Consider the pure state:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

and notice that:

$$CNOT \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

which is the Bell state, which we know is entangled. Hence $CNOT$ is entangling.