20201006 Lecture 3 Pablo S Ocal

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Summary on Lie algebras and their representations.

Definition:

Preliminaries

A Lie algebra is an algebra L with a multiplication $[?,?] : L\times L \longrightarrow L$ satisfying:

(1) Skew-symmetry: $[x,x]=0$ for all $x\in L$,

(2) Jacobi identity: $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ for all $x,y,z\in L$.

Example:

(i) The general linear Lie algebra $\mathfrak{gl}_n(k)$ are all $n\times n$ matrices over k with bracket:

$$[M,N]:=MN-NM \quad \text{for all } M,N\in \mathfrak{gl}_n(k)$$

(ii) The special linear Lie algebra $\mathfrak{sl}_n(k)$ are all $n\times n$ matrices over k with zero trace:

$$\mathfrak{sl}_n(k):=\{M\in\mathfrak{gl}_n(k) \ \ tr(M)=0\}\subseteq \mathfrak{gl}_n(k)$$ It is a Lie subalgebra of $\mathfrak{gl}_n(k)$ with the bracket that it inherits.

(iii) Let V be a vector space of dimension n over a field k, let $L=End(V)$ be the linear endomorphisms of V. Then L is an associative algebra with multiplication the composition of functions. Taking L with bracket:

$$[f,g]:=fg-gf \text{ for all } f,g\in L$$ gives the Lie algebra $\mathfrak{gl}(V)$.

Remark: $\mathfrak{sl}_2=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \,\middle|\, a+d=0\right\}$

Has as $\mathbb{C}$-basis: $h=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, $e=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $f=\begin{bmatrix}0&0\\1&0\end{bmatrix}$.

The brackets are: $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$.

Theorem: There is exactly one irreducible module of L (up to isomorphism) for each dimension.

For each $d\geq 0$, let: $V_d:=Span(x^d, x^{d-1}y,\dots,xy^{d-1},y^d)\subseteq\mathbb{C}[x,y]$, $\dim(V_d)=d+1$,

the subspace of all homogeneous polynomials of degree d, with:

$$\begin{aligned}
\varphi: L &\longrightarrow \mathfrak{gl}(V)\\
e &\longmapsto x\frac{\partial}{\partial y}\\
f &\longmapsto y\frac{\partial}{\partial x}\\
h &\longmapsto x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}
\end{aligned}$$

, i.e. $\varphi(e): V\longrightarrow V$, and extend by linearity.

$x^ay^b \longmapsto bx^{a+1}y^{b-1}$ $(b\geq 1)$

$x^d\longmapsto 0$

$\varphi(f): V\longrightarrow V$,

$x^ay^b\longmapsto ax^{a-1}y^{b+1}$ $(a\geq 1)$

$y^d\longmapsto 0$

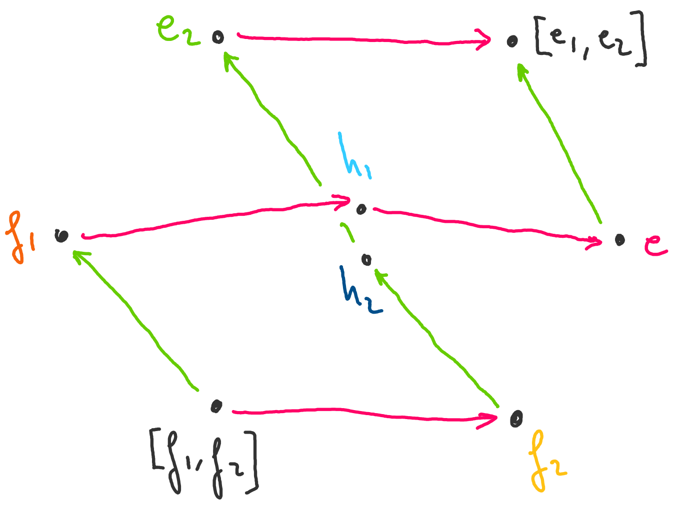
$\varphi(h): V \longrightarrow V$

$\varphi(h): V \longrightarrow V$.
$x^a y^b \longmapsto (a-b)x^a y^b$

Idea: Draw Lie algebras as their adjoint representation.

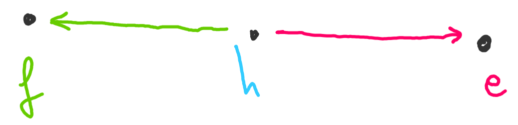
$$\varphi: L \longrightarrow \mathrm{End}(L)$$
$$x \longmapsto \left([x,-]\ L \longrightarrow L,\quad y \longmapsto [x,y] \right)$$

sl_3: has as $\mathbb{C}$-basis. $e_1 = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0\end{bmatrix}$, $e_2 = \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0\end{bmatrix}$, $f_1 = \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0\end{bmatrix}$, $f_2 = \begin{bmatrix} 0&0&0\\0&0&0\\0&1&0\end{bmatrix}$, $h_1 = \begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0\end{bmatrix}$, $h_2 = \begin{bmatrix} 0&0&0\\0&1&0\\0&0&-1\end{bmatrix}$



$\rightarrow$: apply $[e_1,-]$ to the source. (red)
$\rightarrow$: apply $[e_2,-]$ to the source. (green)

sl_2.



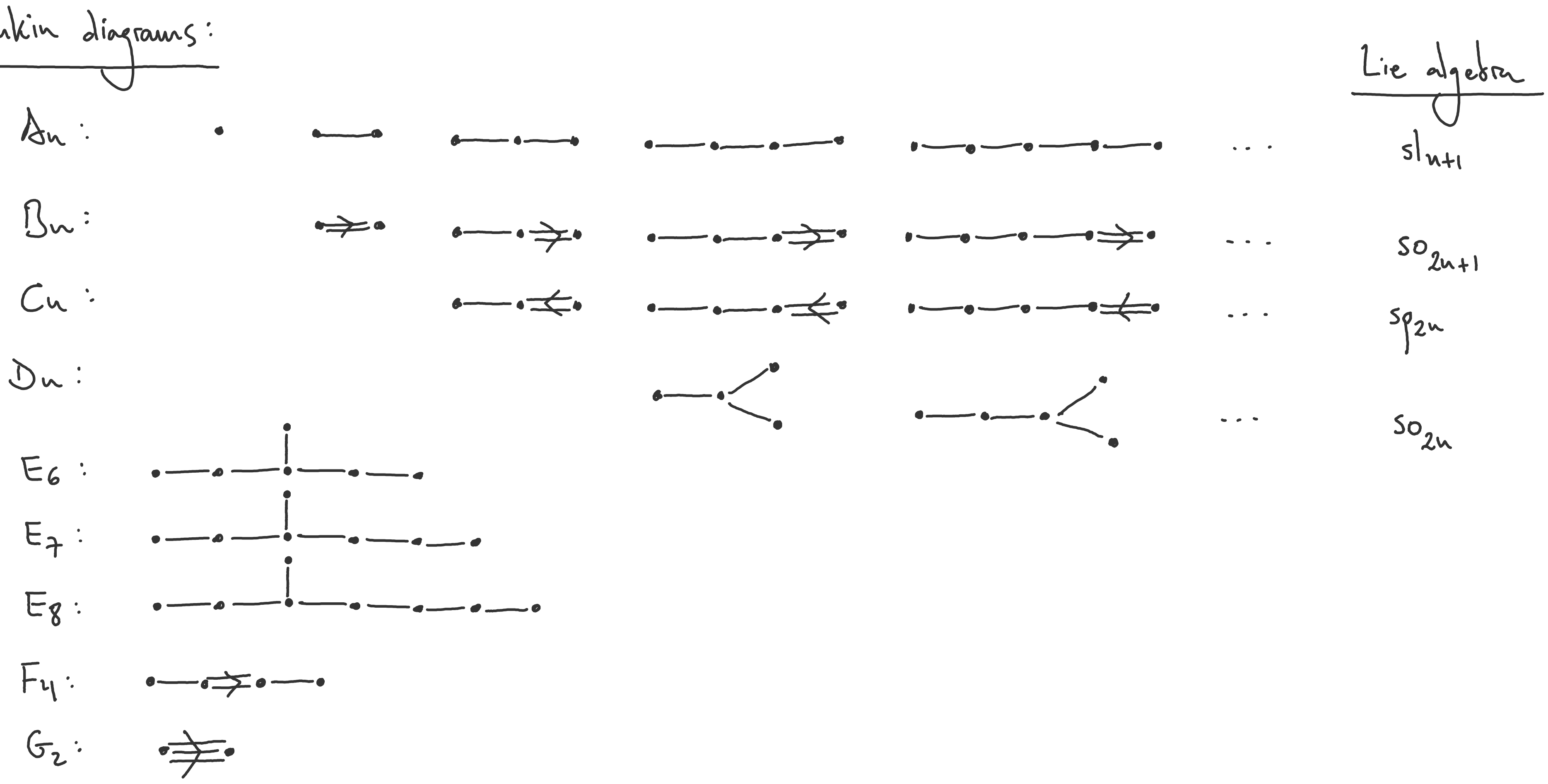
These pictures are called the root system of the Lie algebra.

Morally, the roots are the eigenvectors of the "h" elements. In a general representation, they are called weights

$\boxed{\text{Theorem: The simple finite dimensional Lie algebras are classified by their root systems.}}$

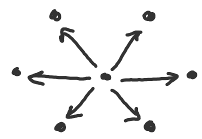
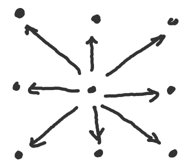
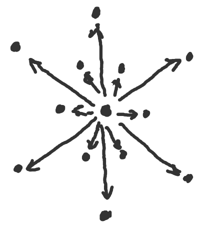
$\boxed{\text{Theorem: The simple finite dimensional Lie algebras are classified by their Dynkin diagrams}}$

Dynkin diagrams:



Remark. The root system of a Dynkin diagram of n nodes lives in n dimensions. This is the rank of the Lie algebra.

Examples.

Dynkin diagram	Lie algebra	Root system
•	sl_2 (A_1)	
•—•	sl_3 (A_2)	
	so_3 (B_2)	
	G_2	

Finding the Lie algebra from the Dynkin diagram.

i j	d_i	a_{ij}	$\langle \alpha_i, \alpha_j \rangle = d_i a_{ij}$
$i=j$		2	$2d_i$
• •		0	0
•—•	$d_i = d_j$	-1	$-d_i$
	2	-1	-2
	3	-1	-3
	1	-2	-2
	1	-3	-3

Remark: a_{ij} give the Cartan matrix.
This tells us how the roots are oriented in the n-dimensional space they live in.
α_i are the simple roots

Presentation of the Lie algebra.

Generators: $E_1, \ldots, E_n, F_1, \ldots, F_n, H_1, \ldots, H_n$.

Relations: $[H_i, H_j] = 0$ $\quad [H_i, E_j] = a_{ij} E_j$ $\quad [E_i, F_j] = 0$ for $i \neq j$, $\quad \underbrace{[E_i, [E_i, \cdots, [E_i, E_j]]]}_{|a_{ij}|+1} = 0$ for $i \neq j$,

$[H_i, F_j] = -a_{ij} \cdot F_j$ $\quad [E_i, F_i] = H_i$ $\quad \underbrace{[F_i, [F_i, \cdots, [F_i, F_j]]]}_{|a_{ij}|+1} = 0$ for $i \neq j$.

Definition: The Borel subalgebra of a Lie algebra L is the subalgebra generated by $E_1, \ldots, E_n, H_1, \ldots, H_n$.
The nilpotent subalgebra of a Lie algebra L is the subalgebra generated by $E_1, \ldots, E_n$.

Definition: Fix λ an element of the weight lattice (i.e. morally an eigenvector of H), say $\lambda = (\lambda_1, \ldots, \lambda_n)$.
A Verma module with highest weight λ is $\quad M_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$

Here $\mathfrak{g}$ is our Lie algebra, $\mathfrak{b}$ its Borel subalgebra, and $\mathbb{C}_\lambda$ the one-dimensional vector space $\mathbb{C}$ with $\mathfrak{b}$-module structure given by H_i acting by λ_i and E_i acting by 0. Denote v_λ the basis of $\mathbb{C}_\lambda$, the highest weight vector.

Definition: Let $\mathfrak{g}$ be a Lie algebra. The universal ... $T(\mathfrak{g})$ ... | Preliminaries

... is universal enveloping algebra is: $U\mathfrak{g} := \frac{T(\mathfrak{g})}{I}$
where I is the two sided ideal over $T(\mathfrak{g})$ generated by elements of the form $a\otimes b - b\otimes a - [a,b]$.

Conceptually:

1) The universal enveloping algebra contains the original Lie algebra in such a way that the bracket multiplication in $\mathfrak{g}$ is now the commutator $ab-ba$ in $U\mathfrak{g}$.
2) The universal enveloping algebra preserves the representation theory: the Lie representations of a Lie algebra $\mathfrak{g}$ correspond to the associative representations of the associative algebra $U\mathfrak{g}$.
In fact, the category of all representations of $\mathfrak{g}$ is isomorphic to the category of left modules over $U\mathfrak{g}$, as abelian categories.

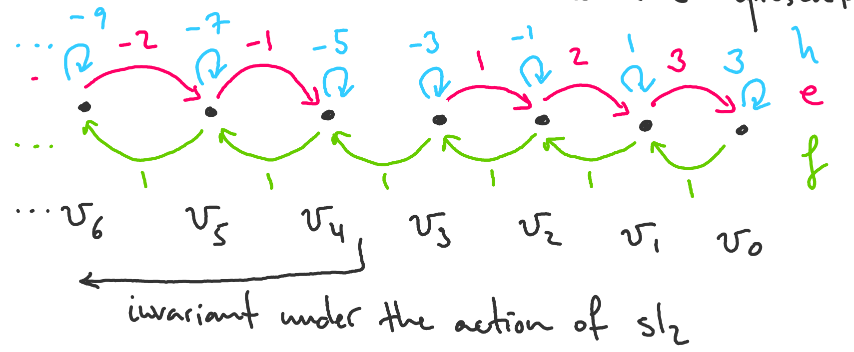
Example: Verma modules for $\mathfrak{sl}_2$.

The Verma module of highest weight λ is spanned by linearly independent vectors: $v_0, v_1, v_2, ..$ and the action of the basis elements is:

$$f v_j = v_{j+1}, \quad e v_j = j(\lambda-(j-1))v_{j-1}, \quad h\cdot v_j = (\lambda - 2j)v_j$$

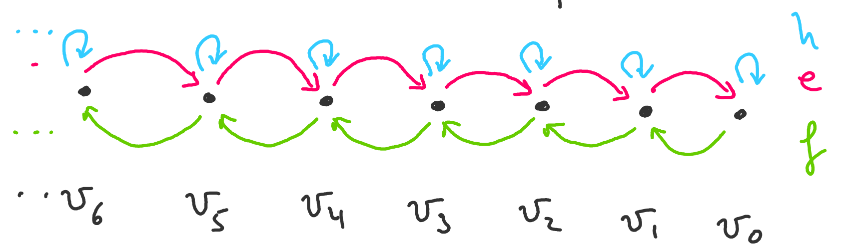
In particular: $hv_0 = \lambda v_0$ and $ev_0 = 0$.

If $\lambda \in \mathbb{N}$ then $ev_{\lambda+1} = 0$ so the span of: $v_{\lambda+1}, v_{\lambda+2}, \dots$ is invariant, so M_λ has a submodule L_λ. The quotient module is the finite dimensional irreducible representation of $\mathfrak{sl}_2$ of dimension $\lambda+1$.



$\lambda = 3$

If $\lambda \in \mathbb{C}\setminus\mathbb{N}$ then M_λ is a simple infinite dimensional representation of $\mathfrak{sl}_2$



$\lambda = \sqrt{-1}$

Definition Let M_λ be a Verma module of highest weight λ. If M_λ has a proper maximal submodule N, we set:

$L_\lambda := M_\lambda / N$ Otherwise, set $L_\lambda := M_\lambda$

Theorem: Verma modules contain at most a unique proper maximal submodule

The quotient L_λ is the unique irreducible representation with highest weight λ. If λ is dominant and integral then this quotient is finite dimensional.

The module M_λ is irreducible if and only if $\lambda = (\lambda_1, .., \lambda_n)$ with $\lambda_1, \dots, \lambda_n \notin \mathbb{N}$