A Reading on “Algebraic Coherent Sheaves” by J.-P. Serre

Pablo Sánchez Ocal
Supervised by Professor Bruno Kahn

July 2016

Université Pierre et Marie Curie

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Acknowledgments

Before starting the reading, I would like to express my deep appreciation and indebtedness to above all Professor Bruno Khan for giving me the opportunity of realizing this dissertation under his supervision.

I acknowledge financial support from La Fundación Mutua Madrileña during this academic year 2015-2016, which has allowed me to study the Master to which this dissertation aims to give completion.

I acknowledge housing in the Colegio de España, the place where most of the work necessary to write and understand the exposed in this dissertation has taken place.

Finally, I would like to thank my parents for all the support they have given me. The completion of this undertaking could not have been possible without them.

1 Introduction

The aim of this dissertation is to present in a concise way several results exposed by Jean-Pierre Serre in [1]. Our main focus will be the study of *algebraic coherent sheaves*, in particular the cohomology of *affine varieties* and *projective varieties* when they take values on the aforementioned sheaves. The motivation of such study is the generalization to abstract algebraic geometry the methods used by H. Cartan, S. Eilenberg and others for the study of functions of several variables and classical algebraic geometry, among others.

An outline of the Sections follows:

Section 2 is meant to be a concise and broad overview of the theory of sheaves that will be used throughout the work. These include **sheaves** and **sheaf cohomology**.

Section 3 covers both **algebraic** and **affine** varieties. For the first general definitions and results are presented, while for the second a complete characterization concerning their cohomology over **algebraic coherent sheaves** and an identification of such sheaves with the cohomology group of degree 0 are established.

Section 4 is where the main interest of the article yields. It contains the definition of **projective** variety and establishes a correspondence of **algebraic coherent sheaves** over them with graduated modules verifying condition (**TF**). This correspondence is bijective when considering the elements in $\mathscr{C}$ and allows the computation of the cohomology.

While the usual notations of $\mathbb{K}$ being a commutative field algebraically closed and $\mathcal{F}$ a sheaf over a topological space X, we will mainly use the notation and reasoning presented by Serre in his article. This notation is somewhat different although equivalent to the one used nowadays, that is, the functorial approach preferred by Grothendieck. For this secondary point of view, we will use as reference the relatively modern text [4].

2 Generalities about sheaves and their cohomology

The definition of a *sheaf* over a topological space X used nowadays is that of a functor over X (or *presheaf*) with some additional structure (see [4]). Equivalently, we have:

Definition 1. *Let X be a topological space. Consider a function $F : X \to \mathcal{F}$ that assigns to every point $x \in X$ an abelian group $\mathcal{F}_x$ and a topology over $\mathcal{F} = \coprod_{x\in X} \mathcal{F}_x$ the sum of those groups seen as sets. We have the usual* projection *$\pi : \mathcal{F} \to X$ by $\pi(f) = x$ when $f \in \mathcal{F}_x$ and the* sum *$\mathcal{F} + \mathcal{F} = \{(f, g) \in \mathcal{F} \times \mathcal{F} | \pi(f) = \pi(g)\}$. A* sheaf of abelian groups *over X is defined by a pair $(F, \mathcal{F})$ under the axioms:*

1. *For every $f \in \mathcal{F}_x$, there are neighborhoods V of f and U of $\pi(f)$ with $\pi(V) \cong U$.*

2. *The maps $- : \mathcal{F} \to \mathcal{F}$ and $+ : \mathcal{F} + \mathcal{F} \to \mathcal{F}$ are continuous.*

Example 1. *Let G be an abelian group. For any topological space X, let $\mathcal{F}_x = G$. We can identify $\mathcal{F} \cong X \times G$ and give it the product topology (using the discrete topology for G), thus obtaining the so called* constant sheaf *isomorphic to G.*

As usual, for opens $U \subset X$ we have (continuous) *sections* $s : U \to \mathcal{F}$ with $\pi \circ s = \mathrm{id}_U$, which form the abelian groups $\Gamma(U, \mathcal{F})$, and homomorphisms $\rho^V_U : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ when $U \subset V$. Moreover, $\mathcal{F}_x = \varinjlim_{x\in U} \Gamma(U, \mathcal{F})$ as a consequence of the first axiom above. The following result states the aforementioned equivalence, that is, we can reduce the study of sheaves to the study of its sections:

Proposition 1. *Let $\mathcal{F}$ be a sheaf of abelian groups. Via the collection $(\Gamma(U, \mathcal{F}), \rho^V_U)$ we can define a sheaf that is canonically isomorphic to $\mathcal{F}$.*

Moreover, the sheaf defined by $(\Gamma(U, \mathcal{F}), \rho^V_U)$ has the additional structure required in [4].In fact, every construction and result in this dissertation can be translated thusly.

Example 2. *Let G be an abelian group, set $F_U = \{f : U \to G | f \text{ map}\}$ for any topological space X. Consider $\rho^V_U : F_V \to F_U$ the restriction map. We obtain a system (F_U, ρ^V_U) and thus the so called* sheaf of germs of functions *$\mathcal{F}$ with values in G, with $F_U \cong \Gamma(U, \mathcal{F})$.*

Given $\mathcal{F}$ a sheaf over X, there are several constructions that can be done. Setting $\mathcal{F}(U) = \pi^{-1}(U) = \coprod_{x\in U} \mathcal{F}_x$ defines the induced sheaf over an open $U \subset X$, and conversely if for an open cover we have a system of compatible sheaves, they define a sheaf over X. Two that deserve a special mention are:

Definition 2. *Let $\mathcal{F}$ be a sheaf over X, we say it is* concentrated *over a closed $Y \subset X$ when $\mathcal{F}_x = 0$ for $x \in X \setminus Y$ (in such cases $\Gamma(X, \mathcal{F}) \cong \Gamma(Y, \mathcal{F}(Y))$). Let $\mathcal{F}(Y)$ be the induced sheaf over Y, by* extending *it by 0 out of Y we obtain a sheaf noted $\mathcal{F}^X$ (which unequivocally determines $\mathcal{F}$).*

Definition 3. *Let $\mathcal{A}$ be a sheaf over X, we say it is a* sheaf of rings *if $\mathcal{A}_x$ is a ring (with unity, varying continuously) for every $x \in X$ and $\cdot : \mathcal{A} + \mathcal{A} \to \mathcal{A}$ is continuous. A sheaf $\mathcal{F}$ is called a* sheaf of $\mathcal{A}$-modules *if F_x is an $\mathcal{A}_x$-module (with unity) and $\cdot : \mathcal{A} + \mathcal{F} \to \mathcal{F}$ is continuous.*

From $\mathcal{F}$, $\mathcal{G}$ sheaves of modules emerge the usual constructions: sub-sheaves and quotient sheaves are well defined and determine the exact sequence $0 \to \Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}/\mathcal{G})$, $\mathcal{A}$-homomorphisms (of sheaves of modules) are given by a collection of homomorphisms of $\mathcal{A}_x$-modules $\varphi_x : F_x \to \mathcal{G}_x$ making $\varphi : \mathcal{F} \to \mathcal{G}$ continuous, the collections $\mathrm{Ker}(\varphi_x)$ and $\mathrm{Im}(\varphi_x)$ define the *kernel* and *image* sheaves of φ, it holds $\mathcal{F}/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$, the *cokernel* sheaf of φ is the quotient $\mathcal{G}/\mathrm{Im}(\varphi)$, the homomorphism φ is *injective* when $\mathrm{Ker}(\varphi) = 0$ and *surjective* when $\mathrm{Coker}(\varphi) = 0$ (*bijective* when both hold), a sequence $\mathcal{I} \to \mathcal{F} \to \mathcal{G}$ is said to be exact when in $\mathcal{F}$ the image coincides with the kernel (in general, every definition concerning homomorphisms of modules has an analogue in homomorphisms of sheaves of modules), the *direct sum* $\mathcal{F} \oplus \mathcal{G}$ is the sheaf of modules defined by the collection $\mathcal{F}_x + \mathcal{G}_x$ as a subset of $\mathcal{F} \times \mathcal{G}$, the *tensor product* $\mathcal{F} \otimes \mathcal{G}$ is the sheaf of modules defined by the collection $\mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$ with structure compatible with the sections over $\mathcal{F}$ and $\mathcal{G}$, the collection $\varphi_x \otimes \phi_x : \mathcal{F}_x \otimes \mathcal{G}_x \to \mathcal{F}'_x \otimes \mathcal{G}'_x$ defines an $\mathcal{A}$-homomorphism $\varphi \otimes \phi : \mathcal{F} \otimes \mathcal{G} \to \mathcal{F}' \otimes \mathcal{G}'$ (again, every property of tensor product of modules has an analogue in tensor product of sheaves of modules), and we finally have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$, the *sheaf of germs of homomorphisms* from $\mathcal{F}$ to $\mathcal{G}$, defined by the family (varying $U \subset X$) of group of homomorphisms from $\mathcal{F}(U)$ to $\mathcal{G}(U)$ (with the analogue results as in modules). This last construction has the particularity that the induced homomorphism $\rho : \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ is not a bijection in general.

Definition 4. *Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules, $s_1, \dots, s_p \in \Gamma(U, \mathcal{F})$. Consider the maps $+_{i=1}^{p} s_i(x) : \mathcal{A}_x^p \to \mathcal{F}_x$, that define $+_{i=1}^{p} s_i : \mathcal{A}(U)^p \to \mathcal{F}(U)$. The* sheaf of relations *among said sections is $\mathrm{Ker}(+_{i=1}^{p} s_i) = \mathcal{R}(s_1, \dots, s_p)$. We say that $\mathcal{F}$ is of* finite type *if it is locally generated by a finite number of sections. We say that $\mathcal{F}$ is* coherent *if: $\mathcal{F}$ is of finite type and $\mathcal{R}(s_1, \dots, s_p)$ is of finite type (over U) for any $s_1, \dots, s_p \in \Gamma(U, F)$. We say that $\mathcal{A}$ is a* coherent sheaf of rings *if it is coherent as an $\mathcal{A}$-module.*

Many operations preserve this structure, one that stands out being:

Theorem 1. *Let $0 \to \mathcal{I} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of sheaves over a topological space X. If any couple among $\mathcal{I}$, $\mathcal{F}$, $\mathcal{G}$ is coherent, then the third is also coherent.*

Proof. See [1]. □

But much more holds: the direct sum (and thus intersection and sum under a bigger sheaf), kernel, cokernel and image of a homomorphism and tensor product, if $\mathcal{A}$ is a coherent sheaf of rings and $\mathcal{F}$ is a coherent sheaf of $\mathcal{A}$-modules, the *annihilator* of $\mathcal{F}$, all produce coherent sheaves. Moreover, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ when $\mathcal{F}$ is coherent and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ is coherent when $\mathcal{G}$ is also coherent. We also have the equivalences:

Theorem 2. *Let $\mathcal{A}$ be a coherent sheaf of rings, $\mathcal{I}$ a coherent sheaf of ideals of $\mathcal{A}$. A sheaf $\mathcal{F}$ of $\mathcal{A}/\mathcal{I}$-modules is $\mathcal{A}/\mathcal{I}$-coherent if and only if $\mathcal{F}$ is $\mathcal{A}$-coherent. In particular, $\mathcal{A}/\mathcal{I}$ is a coherent sheaf of rings.*

Proposition 2. *Let $Y \subset X$ a closed subspace, $\mathcal{A}$ a sheaf of rings over Y. A sheaf $\mathcal{F}$ is $\mathcal{A}$-coherent if and only if $\mathcal{F}^X$ is $\mathcal{A}^X$-coherent.*

Definition 5. *Let* $\mathfrak{U} = \{U_i\}_{i\in I}$ *be a covering of* X, $\mathcal{F}$ *a sheaf. For* $s = (i_0, \dots, i_p) \in I^{p+1}$ *we denote* $U_s = U_{i_0 \dots i_p} = \cap_{j=0}^{p} U_{i_j}$. *A function* $f : I^{p+1} \to \coprod_{s\in I^{p+1}} \Gamma(U_s, \mathcal{F})$ *is called a* p-cochain *of* $\mathfrak{U}$ *with values in* $\mathcal{F}$. *They form the group* $\prod_{s\in I^{p+1}} \Gamma(U_s, \mathcal{F}) = C^p(\mathfrak{U}, \mathcal{F})$, *and this family form the complex* $C(\mathfrak{U}, \mathcal{F})$. *We have the usual* alternating *p-cochains, which form subgroups* $C'^p(\mathfrak{U}, \mathcal{F})$ *and the subcomplex* $C'(\mathfrak{U}, \mathcal{F})$.

The points in I may be viewed as vertexes of a simplex $S(I)$ that defines a complex $K(I)$ and we have the concept of *simplicial endomorphism.* The canonical differential operator $\partial : K_{p+1}(I) \to K_p(I)$ yields a homomorphism $d : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$:

$$(df)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho_j (f_{i_0 \dots \hat{i}_j \dots i_{p+1}}), \text{ with } \rho_j : \Gamma(U_{i_0 \dots \hat{i}_j \dots i_{p+1}}, \mathcal{F}) \to \Gamma(U_{i_0 \dots i_{p+1}}, \mathcal{F}),$$

with $d \circ d = 0$ and the *q-cohomology group* of $C(\mathfrak{U}, \mathcal{F})$ denoted $H^q(\mathfrak{U}, \mathcal{F})$. Since d is stable for alternating cochains, there are $H'^q(\mathfrak{U}, \mathcal{F})$. Moreover, we have $H^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$ and $H'^q(\mathfrak{U}, \mathcal{F}) \cong H^q(\mathfrak{U}, \mathcal{F})$, this being zero for $q > \dim(\mathfrak{U})$.

A refinement $\mathfrak{B}$ of $\mathfrak{U}$ induces a homomorphism $\sigma(\mathfrak{U}, \mathfrak{B}) : H^q(\mathfrak{B}, \mathcal{F}) \to H^q(\mathfrak{U}, \mathcal{F})$. Any $\mathfrak{U}$ is equivalent to certain $\mathfrak{U}'$ indexed by $\mathcal{P}(X)$, thus the coverings can be viewed as an ordered filtrant set. Thus $H^q(\mathfrak{U}, \mathcal{F})$ only depends on the class of $\mathfrak{U}$, and let us define:

$$H^q(X, \mathcal{F}) = \varinjlim_{\mathfrak{U}} H^q(\mathfrak{U}, \mathcal{F}).$$

As before, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. A homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ defines $\varphi^* : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G})$, thus $H^q(X, \cdot)$ is a functor, behaving in the same way as with modules.

Given $0 \to \mathcal{I} \to \mathcal{F} \to \mathcal{G} \to 0$ an exact sequence of sheaves, $0 \to C(\mathfrak{U}, \mathcal{I}) \to C(\mathfrak{U}, \mathcal{F}) \to C(\mathfrak{U}, \mathcal{G})$ is exact, but we do not always have surjectivity. Considering $C_0(\mathfrak{U}, \mathcal{G})$ the image onto $C(\mathfrak{U}, \mathcal{G})$, the constructions above hold and we have the exact sequence $\cdots \to H^q(X, \mathcal{F}) \to H^q_0(X, \mathcal{G}) \to H^{q+1}(X, \mathcal{I}) \to H^{q+1}(X, \mathcal{F}) \to \cdots$, thus $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$ is surjective if $H^1(X, \mathcal{I}) = 0$ and when $H^q_0(X, \mathcal{G}) \cong H^q(X, \mathcal{G})$:

$$\cdots \to H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G}) \to H^{q+1}(X, \mathcal{I}) \to H^{q+1}(X, \mathcal{F}) \to \cdots \text{ is exact.}$$

Considering the induced covering $\mathfrak{U}' = \{Y \cap U_i\}_{i\in I}$ of $Y \subset X$ closed, the restriction induces a homomorphism $\rho^* : H^q(X, \mathcal{F}) \to H^q(Y, \mathcal{F}(Y))$ that is an isomorphism when $\mathcal{F}$ is concentrated over Y, that is $H^q(Y, \mathcal{G}) \cong H^q(X, \mathcal{G}^X)$.

Given $\mathfrak{U}$ and $\mathfrak{B}$ coverings, $s \in S(I)$ and $s' \in S(J)$, set $\mathfrak{U}_{s'} = \{V_{s'} \cap U_i\}_{i\in I}$ and $\mathfrak{B}_s = \{U_s \cap V_j\}_{j\in J}$. We define the double complex $C(\mathfrak{U}, \mathfrak{B}; \mathcal{F})$ as the family $C^{p,q}(\mathfrak{U}, \mathfrak{B}; \mathcal{F}) = \prod_{s\in I^{p+1}, s'\in J^{q+1}} \Gamma(U_s \cap V_{s'}, \mathcal{F})$ with homomorphisms $d_{\mathfrak{U}} : C^{p,q}(\mathfrak{U}, \mathfrak{B}; \mathcal{F}) \to C^{p+1,q}(\mathfrak{U}, \mathfrak{B}; \mathcal{F})$ and $d_{\mathfrak{B}} : C^{p,q}(\mathfrak{U}, \mathfrak{B}; \mathcal{F}) \to C^{p,q+1}(\mathfrak{U}, \mathfrak{B}; \mathcal{F})$ (set $d' = d_{\mathfrak{U}}$, $d'' = (-1)^p d_{\mathfrak{B}}$ and $d = d' + d''$, see [5]). This allow us to determine cases where the induced restriction homomorphisms $H^n(\mathfrak{U}, \mathcal{F}) \to H^n(\mathfrak{U}, \mathfrak{B}; \mathcal{F})$ and $H^n(\mathfrak{B}, \mathcal{F}) \to H^n(\mathfrak{U}, \mathfrak{B}; \mathcal{F})$ are bijections, which yields:

Theorem 3. *Let* $\mathfrak{U}$ *be a covering of a topological space* X, $\mathcal{F}$ *a sheaf over* X. *Suppose there is* $\mathfrak{B}^\alpha$, $\alpha \in A$ *a family of coverings of* X *verifying: every covering* $\mathfrak{W}$ *of* X *has some* $\mathfrak{B}^\alpha$ *as a refinement and* $H^q(B^\alpha_s, \mathcal{F}) = 0$ *for every* $\alpha \in A$, $s \in S(I)$, $q > 0$. *Then* $\sigma(\mathfrak{U}) : H^n(\mathfrak{U}, \mathcal{F}) \to H^n(X, \mathcal{F})$ *is a bijection for every* $n \geq 0$.

3 Algebraic coherent sheaves over affine varieties

We begin by the definitions of *algebraic varieties*, then study the behavior of the *sheaves of local rings* and their cohomology in *affine varieties.*

Definition 6. *Let X be a topological space, we say that it verifies condition (A) if every decreasing sequence of closed subsets of X is stationary.*

Such a space X is compact but does not need to be Hausdorff, every subspace $Y \subset X$ also verifies (A) and if the family $\{Y_i\}_{i=1}^p$ verifies (A), then $\bigcup_{i=1}^p Y_i$ also verifies (A). We must have $X = \bigcup_{i=1}^q V_i$ with all V_i *irreducible* closed subspaces, by asking $V_i \subsetneq V_j$ for any pair (i, j) the V_i are uniquely determined and called the *irreducible components* of X. Given such $X = \bigcup_{i=1}^q V_i$, it is irreducible if and only if all V_i are irreducible and $V_i \cap V_j \neq \emptyset$ for any pair (i, j).

Definition 7. *Let $r \in \mathbb{N}$ and $X = \mathbb{K}^r$ the* affine space *of dimension r over $\mathbb{K}$. We consider the* Zariski topology *over X, where a subset is closed if it is the set of common zeros of a family of polynomials in $\mathbb{K}[X_1, \dots, X_r]$.*

Since $\mathbb{K}[X_1, \dots, X_r]$ is noetherian, X verifies condition (A). It is easily checked that X is irreducible. For any point $x \in X$, we denote $\mathcal{O}_x$ the *local ring* of x (the subring of $\mathbb{K}(X_1, \dots, X_r)$ with denominator non zero in x). An element of $\mathcal{O}_x$ is said *regular* on y in every point $y \in X$ where the denominator is non zero. Thus $\mathcal{O}_x$, $x \in X$, form a subsheaf of rings $\mathcal{O}$ of the sheaf $\mathcal{F}(X)$ of germs of functions over X with values in $\mathbb{K}$.

Let $Y = U \cap V \subset X$ (U open and V closed) be a *locally closed* subspace. For every $x \in Y$ the restriction of a function defines a homomorphism $\varepsilon_x : \mathcal{F}(X)_x \to \mathcal{F}(Y)_x$ whose image $\mathcal{O}_{x,Y} = \varepsilon_x(\mathcal{O}_x)$ are subring of $\mathcal{F}(Y)_x$ that define the subsheaf $\mathcal{O}_Y$ of $\mathcal{F}(Y)$ called *sheaf of local rings* of Y. Given an open $V \subset Y$, a map $f : V \to \mathbb{K}$ is said to be *regular* when in each neighborhood of $x \in V$ it is equal to the restriction to V of a regular rational function in x. The (ring of) sections $\Gamma(V, \mathcal{O}_Y)$ coincide with the regular maps over V. The following is an important result that characterizes the sheaf $\mathcal{O}_Y$:

Corollary 1. *Let $Y = U \cap V$ as above, $I(V) = \{f \in \mathbb{K}[X_1, \dots, X_r] | f(V) = 0\}$, $x \in V$. The ring $\mathcal{O}_{x,Y}$ is isomorphic to the ring of fractions of $\mathbb{K}[X_1, \dots, X_r]/I(V)$ relative to the maximal ideal defined by the point x.*

Given U, V locally closed subspaces of K^r, K^s respectively, we are interested in studying the *regular* maps over U, that is maps $\varphi : U \to V$ that are continuous and for $x \in U$, $f \in \mathcal{O}_{\varphi(x)}, V$, then $f \circ \varphi \in \mathcal{O}_{x,U}$. For a map to be regular is sufficient and enough to be regular in each coordinate, the composite of regular maps is regular and a bijection with regular inverse is called *biregular isomorphism.* Identifying $\mathbb{K}^{r+s}$ to $\mathbb{K}^r \times \mathbb{K}^s$, the Zariski topology is a refinement of the induced topology, and thus $\mathcal{O}_{U \times V}$ is well defined. As expected, the product of two maps $\varphi \times \varphi' : U \times U' \to V \times V'$ is regular if and only if each map is regular, and similarly with isomorphisms.

Definition 8. *An* algebraic variety *over $\mathbb{K}$ is a set X together with a topology and a subsheaf $\mathcal{O}_X$ of the sheaf $\mathcal{F}(X)$ of germs of functions over X with values in $\mathbb{K}$, verifying:*

(VA_I) *There is a finite open cover* $\mathfrak{B} = \{V_i\}_{i\in I}$ *of* X *such that each* $V_i \cong U_i$ *for some locally closed subspace* U_i *of an affine space (both with the induced structure).*

(VA_{II}) *The diagonal* Δ *of* $X \times X$ *is closed in* $X \times X$.

While (VA_I) can be asked, we must justify (VA_{II}). If only (VA_I) is satisfied, we say that X is a *prealgebraic variety*, X automatically satisfies condition (A) via the isomorphisms $\varphi_i : V_i \to U_i$, called *charts* of V_i, the topology over X is called the *Zariski topology* over X and the sheaf $\mathcal{O}_X$ is called the *sheaf of local rings* of X. The structure of prealgebraic variety over $X = \bigcup_{i=1}^p X_i$ can be well defined via the structure of prealgebraic variety of each X_i mwhen they are compatible, making them open in X, and thus if X' is another prealgebraic variety, the set $X \times X'$ has a structure of prealgebraic variety induced by the structures of each component: this allow us to consider the topology of $X \times X$, making (VA_{II}) consistent.

The concept of *regular* maps $\varphi : X \to Y$ among algebraic varieties is analogous to the one among locally closed subspaces of affine spaces, as are the properties. The natural way of defining the *induced* structure of algebraic variety of a locally closed subset $X' \subset X$ works well, and X' is called *subvariety* of X. Similarly, the product $X \times Y$ yields an algebraic variety called *product variety.*

Let X be an irreducible algebraic variety. For a non empty open $U \subset X$, set $\mathcal{A}_U = \Gamma(U, \mathcal{O}_X)$ (which is an integral ring). Denote by $\mathcal{K}_U$ the field of quotients, we have an isomorphism $\mathcal{K}_U \cong \mathcal{K}_V$ when $U \subset V$. The family $\mathcal{K}_U$ defines a *sheaf of fields* $\mathcal{K}$, with $\mathcal{K}_x$ canonically isomorphic to the field of quotients of $\mathcal{O}_{x,X}$. It turns out that $\mathcal{K}$ is a constant sheaf and the sections are $\Gamma(U, \mathcal{K}) \cong \mathcal{K}_x = \mathbb{K}(X)$ for every $x \in X$, a field that we call *field of rational functions* over X. The identification in Corollary 1 means that $\mathbb{K}(X)$ is an extension of finite type of $\mathbb{K}$ (and in particular $\mathcal{O}_{x,X}$ is a subring of $\mathbb{K}(X)$), we say that the degree of transcendence of this extension is the *dimension* of X. For reducible algebraic varieties $X = \bigcup_{i\in I} Y_i$, we set $\dim(X) = \sup_{i\in I}(\dim(Y_i))$.

Proposition 3. *Let* V *be an algebraic variety, then* $\mathcal{O}_V$ *is a coherent sheaf of rings over* V. *In particular for* X *an affine space,* $\mathcal{O}$ *is a coherent sheaf of rings.*

Proof. Consider first an affine space X, let $x \in X$, U neighborhood of x and $f_1, \ldots, f_p \in \Gamma(U, \mathcal{O})$. Suppose we have $y \in U$ and $g_i \in \mathcal{O}_y$ with $\sum_{i=1}^p g_i f_i = 0$ locally. We may write as a quotient of polynomials: $f_i = P_i/Q$ and $g_i = R_i/Q$, thus equivalently $\sum_{i=1}^p R_i P_i = 0$. Since the ring of polynomials is noetherian, the module of relations among the P_i is of finite type, and thus $\mathcal{R}(f_1, \ldots, f_p)$ is of finite type.

For the algebraic variety, the property is local, so we may take V as a locally closed subvariety of an affine space X. Considering $\mathcal{I}(V)$ the sheaf of ideals of the functions that are locally zero restricted to V, it is readily checked that it is a coherent sheaf of $\mathcal{O}$-modules, thus $\mathcal{O}/\mathcal{I}(V) = \mathcal{O}_V^X$, a coherent sheaf of rings over X that is zero out of V and $\mathcal{O}_V$ inside V. Thus $\mathcal{O}_V$ is a coherent sheaf of rings over V by Proposition 2. □

Definition 9. *Let* V *be an algebraic variety, we say that* $\mathcal{F}$ *is an* algebraic sheaf *over* V *if it is a sheaf of* $\mathcal{O}_V$*-modules. For* $\mathcal{F}$, $\mathcal{G}$ *algebraic sheaves, a* $\mathcal{O}_V$*-homomorphism*

$\varphi : \mathcal{F} \to \mathcal{G}$ *is called an* algebraic homomorphism. *An algebraic sheaf* $\mathcal{F}$ *is said to be* coherent *if it is a coherent sheaf of* $\mathcal{O}_V$*-modules.*

Proposition 4. *Let* W *be a closed subvariety of an algebraic variety* V*. If* $\mathcal{F}$ *is an algebraic coherent sheaf over* W*, then* $\mathcal{F}^V$ *is an algebraic coherent sheaf over* V*. Conversely, if* $\mathcal{G}$ *is an algebraic coherent sheaf over* V *with annihilator containing* $\mathcal{I}(W)$*, then* $\mathcal{G}(W)$ *is an algebraic coherent sheaf over* W*.*

In particular, every algebraic coherent sheaf over an affine variety can be viewed as an algebraic coherent sheaf over the affine space. Remark that when $\mathcal{G}$ is zero outside of W, we can only guarantee that a power of $\mathcal{I}(W)$ is contained in the annihilator of $\mathcal{G}$.

Definition 10. *Let* V *be an algebraic variety. we say that* V *is* affine *if it is isomorphic to a closed subvariety of an affine space. An open subset* U *of an algebraic variety* V *is called* affine *if provided with the structure of algebraic variety induced by* V*, is an affine variety.*

The product of affine varieties and closed subvarieties of an affine variety remain affine. Moreover, the intersection of affine opens is affine, and given f a regular map over V an affine variety, the subset $V_f = \{x \in V | f(x) \neq 0\}$ is an affine open. Given V a closed subvariety of an affine space, $T \subset V$ closed, the family of affine opens V_P with polynomials $P(T) = 0$ form a basis for the topology of $V \setminus T$. Thus, the family of affine open subsets of an algebraic variety X form a basis of its topology, and the coverings of X given by affine subsets are arbitrarily refined.

Some technical results about irreducible algebraic varieties are of interest when studying the cohomology of sheaves over varieties. Let V be a closed subvariety of $\mathbb{K}^r$, $I(V)$ the ideal of $\mathbb{K}[X_1, \ldots, X_r]$ of polynomials that are zero over V. There is a canonical injective homomorphism $\iota : \mathbb{K}[X_1, \ldots, X_r]/I(V) \to \Gamma(V, \mathcal{O}_V)$ that is bijective when V is irreducible. Suppose X is an irreducible algebraic variety, Q a regular function over X and P a regular function over X_Q, then for n big enough the rational function $Q^n P$ is regular over X. If we also have an algebraic coherent sheaf $\mathcal{F}$ over X and $s \in \Gamma(X, \mathcal{F})$ with $s(X_Q) = 0$, then for n big enough $Q^n s$ is zero over X. The way of proving these facts is to use the locality of the properties to be verified and reasoning on fractions of polynomials (see [6] or [7] for the background in commutative algebra).

Proposition 5. *Let* X *be an irreducible affine variety,* Q_i*,* $i \in I$*, a finite family of regular functions over* X *non zero simultaneously and* $\mathfrak{U} = \{X_{Q_i}\}_{i \in I}$*. If* $\mathcal{F}$ *is an algebraic coherent subsheaf of* $\mathcal{O}_X^p$*, then* $H^q(\mathfrak{U}, \mathcal{F}) = 0$ *for every* $q > 0$*.*

Proof. We shall only say that the explicit forms of cocycles and fractions of polynomials are used: it is mainly routine computations helped by the technical results above. □

As a consequence of this result and that the coverings used are arbitrarily refined, we have that $H^q(X, \mathcal{F}) = 0$ for every $q > 0$. In fact, this implies that the exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{O}_X^p \to \mathcal{O}_X^p/\mathcal{F} \to 0$ induces and exact sequence $0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{O}_X^p) \to \Gamma(X, \mathcal{O}_X^p/\mathcal{F}) \to 0$ and that $\mathbb{K}[X_1, \ldots, X_r]/I(V) \cong \Gamma(V, \mathcal{O}_V)$ by the homomorphism ι above.

Theorem 4. *Let $\mathcal{F}$ be an algebraic coherent sheaf over an affine variety X. For every $x \in X$, $\mathcal{F}_x$ is generated as $\mathcal{O}_{x,X}$-module by the global sections $\Gamma(X, \mathcal{F})$.*

Proof. We shall only sketch it. Since X is affine, we may extend it as a closed subvariety of an affine space $\mathbb{K}^r$ and thus extend $\mathcal{F}$ onto $\mathcal{F}^X$ which remains an algebraic coherent sheaf, as noticed. It is thus enough to consider $X = \mathbb{K}^r$. Locally $\mathcal{F}$ is isomorphic to a quotient of a sheaf $\mathcal{O}^p$, thus there are a finite number of polynomials Q_i (non zero and not simultaneously zero) that determine surjections $\varphi_i : \mathcal{O}^{p_i} \to \mathcal{F}$ over X_{Q_i}. Let $x \in U_0$, $\mathcal{F}_x$ is generated by the sections $\Gamma(U_0, \mathcal{F})$ and Q_0 is invertible in $\mathcal{O}_x$.

It is thus enough to prove that if $s_0 \in \Gamma(U_0, \mathcal{F})$, then there is $N \in \mathbb{N}$ and $s \in \Gamma(X, \mathcal{F})$ with $s = Q_0^N s_0$ over U_0. Using the technical results above, for n big enough we can find a family of sections $s'_i \in \Gamma(U_i, \mathcal{F})$ that coincide with $Q_0^n s_0$ over $U_i \cap U_0$, for m big enough we have $Q_0^m(s'_i - s'_j) = 0$ over $U_i \cap U_j$, and thus the sections $Q_0^m s'_i$ are compatible and determine a unique section $s \in \Gamma(X, \mathcal{F})$ with $s = Q_0^{n+m} s_0$ over U_0, as desired. □

This means that the sheaf $\mathcal{F}$ is isomorphic to a quotient sheaf of a sheaf $\mathcal{O}_X^p$, since X is compact and any sections generating $\mathcal{F}_x$ also generate locally. In fact, an exact sequence $\mathcal{I} \to \mathcal{F} \to \mathcal{G}$ of algebraic coherent sheaves over an affine variety X determine an exact sequence $\Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$.

Theorem 5. *Let X be an affine variety, Q_i, $i \in I$, a finite family of regular functions over X non zero simultaneously and $\mathfrak{U} = \{X_{Q_i}\}_{i \in I}$. If $\mathcal{F}$ is an algebraic coherent sheaf over X, then $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for every $q > 0$.*

Proof. We shall only sketch it. Suppose first X irreducible, we can find an exact sequence of sheaves $0 \to \mathcal{R} \to \mathcal{O}_X^p \to \mathcal{F} \to 0$ that induces an exact sequence of complexes $0 \to C(\mathfrak{U}, \mathcal{R}) \to C(\mathfrak{U}, \mathcal{O}_X^p) \to C(\mathfrak{U}, \mathcal{F}) \to 0$ that induces an exact sequence in cohomology $\cdots \to H^q(\mathfrak{U}, \mathcal{O}_X^p) \to H^q(\mathfrak{U}, \mathcal{F}) \to H^{q+1}(\mathfrak{U}, \mathcal{R}) \to \cdots$, with $H^q(\mathfrak{U}, \mathcal{O}_X^p) = 0 = H^{q+1}(\mathfrak{U}, \mathcal{R})$ for every $q > 0$ by Proposition 5, thus $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for every $q > 0$.

Let $X \subset \mathbb{K}^r$ be any affine variety. The extension $\mathcal{F}^{\mathbb{K}^r}$ is an algebraic coherent sheaf, Q_i are induced by polynomials P_i and $I(X)$ is generated by some R_j. Since P_i, R_j are not zero simultaneously over $\mathbb{K}^r$, they induce a covering $\mathfrak{U}'$ of $\mathbb{K}^r$ and $H^q(\mathfrak{U}', \mathcal{F}^{\mathbb{K}^r}) = 0$ for $q > 0$. Since $C(\mathfrak{U}', \mathcal{F}^{K^r}) \cong C(\mathfrak{U}, \mathcal{F})$ as complexes, $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for $q > 0$. □

Since the coverings used are arbitrarily refined, this technicality implies:

Corollary 2. *Let X be an affine variety and $\mathcal{F}$ an algebraic coherent sheaf over X. Then $H^q(X, \mathcal{F}) = 0$ for every $q > 0$.*

Moreover, the exact sequence of sheaves $0 \to \mathcal{I} \to \mathcal{F} \to \mathcal{G} \to 0$ induces and exact sequence $0 \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to 0$. The above also implies that $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for any finite affine cover $\mathfrak{U}$ of X, and building on this and using Theorem 3 we have:

Theorem 6. *Let X be an algebraic variety, $\mathfrak{U}$ a finite affine covering of X. If $\mathcal{F}$ is an algebraic coherent sheaf over X, then the homomorphisms $\sigma(\mathfrak{U}) : H^n(\mathfrak{U}, \mathcal{F}) \to H^n(X, \mathcal{F})$*

are bijections for every $n \geq 0$. If we have $0 \to \mathcal{I} \to \mathcal{F} \to \mathcal{G} \to 0$ an exact sequence of sheaves over X with $\mathcal{I}$ being an algebraic coherent sheaf, then the homomorphism $H_0^q(\mathfrak{U}, G) \to H^q(\mathfrak{U}, \mathcal{G})$ is a bijection for every $q \geq 0$.

This means that the homomorphism $H_0^q(X, G) \to H^q(X, \mathcal{G})$ is a bijection for every $q \geq 0$, thus $\cdots \to H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G}) \to H^{q+1}(X, \mathcal{I}) \to H^{q+1}(X, \mathcal{F}) \to \cdots$ is an exact sequence.

Definition 11. *Let V be an affine variety with sheaf of local rings $\mathcal{O}$. The ring $A = \Gamma(V, \mathcal{O})$ is called the* coordinate ring *of V.*

We may see A as an algebra over $\mathbb{K}$ without nilpotent elements different from zero, when $V \subset \mathbb{K}^r$ we identify $A \cong \mathbb{K}[X_1, \ldots, X_r]/I(V)$ and thus A is finitely generated. Conversely, let A be a commutative $\mathbb{K}$-algebra without nilpotent elements different from zero and finitely generated, then there is an affine variety V (determined up to isomorphism) with $\Gamma(V, \mathcal{O}) \cong A$.

Let M be an A-module, identify it with the constant sheaf that defines over V and consider it as a sheaf of A-modules. Consider $\mathcal{O}$ a sheaf of A-modules. Set $\mathcal{A}(M) = \mathcal{O} \otimes_A M$, an algebraic sheaf over V. The assignment $\mathcal{A}(M)$ is functorial, since $\varphi : M \to M'$ an A-homomorphism defines $\mathcal{A}(\varphi) = 1 \otimes \varphi : \mathcal{A}(M) \to \mathcal{A}(M')$ a homomorphism. This functor is exact, $\mathcal{A}(M) = 0$ implies $M = 0$ and if M is of finite type, then $\mathcal{A}(M)$ is an algebraic coherent sheaf.

Let $\mathcal{F}$ be an algebraic sheaf over V, consider the global sections $\Gamma(\mathcal{F}) = \Gamma(V, \mathcal{F})$, which have a structure of A-module. This assignment is functorial, since $\varphi : \mathcal{F} \to \mathcal{G}$ an algebraic homomorphism defines $\Gamma(\varphi) : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{G})$ an A-homomorphism. This functor is also exact when applied to algebraic coherent sheaves by Theorem 4 and if $\mathcal{F}$ is coherent, then $\Gamma(\mathcal{F})$ is of finite type.

Theorem 7. *The functors $\mathcal{A}(\cdot)$ and $\Gamma(\cdot)$ are quasi-inverse:*

1. *If M is an A-module of finite type, then $\Gamma(\mathcal{A}(M)) \cong M$ canonically.*
2. *If $\mathcal{F}$ is an algebraic coherent sheaf over V, then $\mathcal{A}(\Gamma(\mathcal{F})) \cong \mathcal{F}$ canonically.*

This correspondence can be extended to projective modules (see [5] or [8]), since for M an A-module of finite type, M is projective if and only if $\mathcal{O}_x \otimes_A M$ is free as an $\mathcal{O}_x$-module for every $x \in V$. Since having for an algebraic coherent sheaf $\mathcal{F}$ over V that $\mathcal{F}_x$ is isomorphic to $\mathcal{O}^p$ for every $x \in V$ means that $\mathcal{F}$ is locally isomorphic to $\mathcal{O}^p$, with $p \in N$ constant over the connected components of V, this yields:

Corollary 3. *Let $\mathcal{F}$ be an algebraic coherent sheaf over a connected affine variety V. Then $\Gamma(\mathcal{F})$ is a projective A-module if and only if $\mathcal{F}$ is locally isomorphic to a sheaf $\mathcal{O}^p$.*

4 Algebraic coherent sheaves over projective varieties

We begin by the definition of *projective space* and proceed to identify *coherent sheaves* over them with graduated modules, which enables the study of their cohomology.

Definition 12. *Let $r \in \mathbb{N}$ and $Y = \mathbb{K}^{r+1} \setminus \{0\}$ with the componentwise multiplication by elements of the multiplicative group $\mathbb{K}^\times$. We define $X = \mathbb{P}_r(\mathbb{K})$ the* projective space of dimension r *as the quotient of Y by the equivalence relation $y' \sim y$ when there is $\lambda \in \mathbb{K}^\times$ with $(y'_0, \dots, y'_r) = \lambda(y_0, \dots, y_r)$, and denote π the canonical projection.*

For each $i = 0, \dots, n$ we denote $t_i : \mathbb{K}^{r+1} \to \mathbb{K}$ the projections onto the coordinates, $V_i = \{x \in \mathbb{K}^{r+1} | t_i(x) \neq 0\}$ and $U_i = \pi(V_i)$. We have $\mathfrak{U} = \{U_i\}_{i=0}^r$ a covering of X, for every $i, j = 0, \dots, n$ the function t_j/t_i is regular over V_i, invariant by $\mathbb{K}^\times$ and thus defines a function over U_i. Fixing i and varying j, these functions define a bijection $\psi_i : U_i \to \mathbb{K}^r$.

Since $\mathbb{K}^{r+1}$ has a structure of algebraic variety, Y has the induced structure. Letting X have the quotient topology, for an open $U \subset X$ set $A_U = \Gamma(\pi^{-1}(U), \mathcal{O}_Y)$ and $A^0_U \subset A_U$ the elements invariant by $\mathbb{K}^\times$. When $U \subset V$ there is a restriction homomorphism $\varphi^V_U : A^0_V \to A^0_U$, and thus the family (A^0_U, φ^V_U) defines the sheaf denoted $\mathcal{O}_X$, which may be considered as a subsheaf of $\mathcal{F}(X)$ the sheaf of germs of functions over X. Thus $f \in \mathcal{O}_{x,X}$ if and only if it can be written locally as P/Q with $P, Q \in \mathbb{K}[t_0, \dots, t_r]$ homogeneous of the same degree and $Q(y) \neq 0$ whenever $y \in \pi^{-1}(x)$, denoted $Q(x) \neq 0$.

Proposition 6. *The projective space X with $\mathcal{O}_X$ as above is an algebraic variety.*

As with affine varieties, we define:

Definition 13. *An algebraic variety V will be said to be a* projective variety *if it is isomorphic to a closed subvariety of the projective space X.*

Applying Theorem 6 it immediately follows that for $\mathcal{F}$ an algebraic coherent sheaf over X, then $H^n(X, \mathcal{F}) \cong H^n(\mathfrak{U}, \mathcal{F})$ for $n \in \mathbb{N}$ and $H^n(X, \mathcal{F}) = 0$ for $n > r$. This can be generalised: let V an algebraic variety isomorphic to a locally closed subvariety of X and $\mathcal{F}$ an algebraic coherent sheaf over V such that $\mathcal{F}(V \setminus W) = 0$ for some subvariety $W \subset V$, then $H^n(V, \mathcal{F}) = 0$ for $n > \dim(W)$. To establish this we need two technical results: if $k = \dim(W)$ there are $k+1$ homogeneous polynomials $P_i(t_0, \dots, t_r)$ of non trivial degree that are zero over $X \setminus U$ (for U open with $V \subset U$ closed) but not zero simultaneously over W, and if $P(t_0, \dots, t_r)$ is an homogeneous polynomial of non trivial degree, then $X_P = \{x \in X | P(x) \neq 0\}$ is an affine open.

Definition 14. *Let $\mathcal{F}$ be an algebraic sheaf over X, $n \in \mathbb{Z}$. Consider $\mathcal{F}_i = \mathcal{F}(U_i)$, we have isomorphisms $\theta_{ij}(n) = \cdot t^n_j / t^n_i : \mathcal{F}_j(U_i \cap U_j) \to \mathcal{F}_i(U_i \cap U_j)$ and $\theta_{ij}(n) \circ \theta_{jk}(n) = \theta_{ik}(n)$ in $U_i \cap U_j \cap U_k$ (they are compatible). This defines $\mathcal{F}(n)$ an algebraic sheaf over X.*

This construction sets canonical isomorphisms $\mathcal{F}(0) \cong \mathcal{F}$ and $\mathcal{F}(n)(m) \cong \mathcal{F}(n+m)$, $\mathcal{F}(n)$ is locally isomorphic to $\mathcal{F}$ and thus an exact sequence of sheaves $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ gives an exact sequence of sheaves $\mathcal{F}(n) \to \mathcal{F}'(n) \to \mathcal{F}''(n)$.

Considering $\mathcal{O}$, we obtain the sheaf $\mathcal{O}(n)$ that can be canonically identified with the sheaf $\mathcal{O}'(n)$ defined by the family A_U^n of homogeneous elements of degree n of A_U (in particular $f \in \mathcal{O}'(n)_x$ if and only if it can be written locally as P/Q with $P, Q \in \mathbb{K}[t_0, \ldots, t_r]$ homogeneous with $Q(x) \neq 0$ and $\deg(P) - \deg(Q) = n$). Moreover, given an algebraic sheaf $\mathcal{F}$, the construction gives a canonical isomorphism $\mathcal{F}(n) \cong \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(n)$.

Some analogous technicalities to those exposed in Page 10 arise. Let V be an affine variety, $\mathcal{F}$ an algebraic coherent sheaf over V, Q a regular function over V and $s \in \Gamma(V_Q, \mathcal{F})$, then for $n \in \mathbb{N}$ big enough there is $s' \in \Gamma(V, \mathcal{F})$ with $s' = Q^n s$ over V_Q. Let $s_i \in \Gamma(U_i, \mathcal{F})$, then for $n \in \mathbb{N}$ big enough there is $s' = (s'_j)_{j=0}^r \in \Gamma(X, \mathcal{F}(n))$ with $s'_i = s_i$. These allow us to prove analogous results to those of Theorem 4, that is, the sections of $\mathcal{F}(n)$ behave in an analogous way as the sections of $\mathcal{F}$ whenever it is coherent:

Theorem 8. *Let $\mathcal{F}$ an algebraic coherent sheaf over X. There exists $n(\mathcal{F}) \in \mathbb{Z}$ such that for every $n \geq n(\mathcal{F})$ and every $x \in X$, $\mathcal{F}(n)_x$ is generated as $\mathcal{O}_x$-module by the global sections $\Gamma(X, \mathcal{F}(n))$.*

Proof. We shall only sketch it. A section $s \in \Gamma(X, \mathcal{F}(n))$ is by definition a system $s = (s_i)_{i \in I}$ of sections $s_i \in \Gamma(U_i, \mathcal{F})$, verifying the compatibility conditions $s_i = (t_j^n / t_i^n) s_j$ over $U_i \cap U_j$. Since $U_i \cong \mathbb{K}^r$ for every $i \in I$, as a result of Theorem 4 there are a finite number of sections $s_i^a \in \Gamma(U_i, \mathcal{F})$ with $a \in A$ that generate $\mathcal{F}_x$ for every $x \in U_i$. As we have pointed before as a technicality, there is $n \in \mathbb{N}$ big enough for which we can find sections $s'^a \in \Gamma(X, \mathcal{F}(n))$ with $s_i'^a = s_i^a$. Both indexes I and A being finite, we can fix n for every $i \in I$ and $a \in A$. Thus, $\Gamma(X, \mathcal{F}(n))$ generate $\mathcal{F}(n)_x$ for every $x \in X$. $\square$

Continuing the analogy, the sheaf $\mathcal{F}$ is isomorphic to a quotient sheaf of $\mathcal{O}(n)^p$, since $\mathcal{F}(-n)_x$ is generated by $\Gamma(X, \mathcal{F}(-n))$ and thus $\mathcal{F}(-n)$ is isomorphic to a quotient of $\mathcal{O}^p$ thus $\mathcal{F} \cong \mathcal{F}(-n)(n)$ is isomorphic to a quotient of $\mathcal{O}(n)^p \cong \mathcal{O}^p(n)$.

Definition 15. *Set $S = \mathbb{K}[t_0, \ldots, t_r]$ for $r \in \mathbb{N}$ a graduated algebra, $S = \bigoplus_{n \in \mathbb{Z}} S_n$ with S_n the vector space of homogeneous polynomials of degree n (setting $S_n = 0$ for $n < 0$).*

For $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a *graduated module* over S, we say an element of M_n has *degree* n and $N = \bigoplus_{n \in \mathbb{Z}} N \cap M_n$ is an *homogeneous* submodule of M. For M, M' graduated S-modules, an S-homomorphism $\varphi : M \to M'$ is said to be *homogeneous of degree* s when $\varphi(M_n) \subset M'_{n+s}$. Given $n \in \mathbb{Z}$, we denote $M(n) = \bigoplus_{m \in \mathbb{Z}} M(n)_m = \bigoplus_{m \in \mathbb{Z}} M_{n+m}$. A graduated S-module L is said to be of *finite type* if it is generated by a finite number of elements, *free* if it admits a basis consisting of homogeneous elements (thus $L = \bigoplus_{i \in I} S(n_i)$), *free of finite type* if its basis is finite (thus I is finite).

As in [2]: the *class* of graduated S-modules M having $M_n = 0$ for n big enough will be denoted $\mathscr{C}$, for an exact sequence by homomorphisms of degree 0 of graduated S-modules $A \to B \to C$ having A, $B \in \mathscr{C}$ imply $C \in \mathscr{C}$, an homomorphism $\varphi : A \to B$ is said *injective*, *surjective* or *bijective* when $\operatorname{Ker}(\varphi) \in \mathscr{C}$, $\operatorname{Coker}(\varphi) \in \mathscr{C}$ or both, respectively.

Definition 16. *A graduated S-module M is said to* verify condition (TF) *if there is $m \in \mathbb{Z}$ such that $\bigoplus_{n \geq m} M_n$ is of finite type, equivalently M is $\mathscr{C}$-isomorphic to a module of finite type. In particular, the modules verifying (TF) form a class containing $\mathscr{C}$.*

We remark that the following constructions are analogous to what was done in Page 12, although then for $\mathcal{A}(M)$ we had a concise way to express them by a tensor product, the only differences being that here we only consider the homogeneous component of degree 0, and for $\Gamma(\mathcal{F})$ we only needed one ring of sections, when now we need many.

Let $U \in \mathcal{P}(X)$ not empty, denote $S(U) = \bigcup_{n\in\mathbb{Z}}\{P \in S_n | P(x) \neq 0 \,\forall x \in U\}$, writing $S(x)$ instead of $S(\{x\})$. For M a graduated S-module, set $M_U = \bigcup_{n\in\mathbb{Z}}\{m/Q | m \in M_n,\, Q \in S(U),\, \deg(Q) = n\}$, writing M_x instead of $M_{\{x\}}$. We determine an equivalence relation by identifying m/Q, $m'/Q' \in M_U$ if there is $Q'' \in S(U)$ with $Q''(Q'm - Qm') = 0$. In particular for S, we have S_U the ring of rational fractions P/Q of homogeneous polynomials with $\deg(P) = \deg(Q)$ and $Q \in S(U)$. Thus we can define over M_U a structure of S_U-module by setting $m/Q + m'/Q' = (Q'm + Qm')/QQ'$ and $(P/Q)(m/Q') = Pm/QQ'$. Since $U \subset V$ means $S(V) \subset S(U)$, we have the restriction homomorphisms $\varphi_U^V : M_V \to M_U$ and thus considering only open subsets, the system (M_U, φ_U^V) defines the sheaf $\mathcal{A}(M)$. We clearly have $\mathcal{A}(M)_x = \varinjlim_{x\in U} M_U = M_x$, and in particular $\mathcal{A}(S) = \mathcal{O}$. It follows that $\mathcal{A}(M)$ is a sheaf of $\mathcal{A}(S)$-modules, thus an algebraic sheaf over X. The assignment $\mathcal{A}(M)$ is functorial, since $\varphi : M \to M'$ an S-homomorphism defines $\varphi_U : M_U \to M'_U$ an S_U-homomorphism and thus $\mathcal{A}(\varphi) = \mathcal{A}(M) \to \mathcal{A}(M')$ a homomorphism of sheaves. This functor is additive and exact, $\mathcal{A}(M)(n) \cong \mathcal{A}(M(n))$ for $n \in \mathbb{Z}$, and when M verifies condition (TF) then $\mathcal{A}(M)$ is coherent and $\mathcal{A}(M) = 0$ if and only if $M \in \mathscr{C}$. In fact, when M and M' are graduated S-modules verifying condition (TF), a homomorphism $\varphi : M \to M'$ is injective, surjective or bijective if and only if $\mathcal{A}(\varphi) : \mathcal{A}(M) \to \mathcal{A}(M')$ is $\mathscr{C}$-injective, $\mathscr{C}$-surjective or $\mathscr{C}$-bijective, respectively.

Let $\mathcal{F}$ be an algebraic sheaf over X, set $\Gamma(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathcal{F})_n = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$, a graduated group. For $s \in \Gamma(X, \mathcal{F}(q))$, $P \in S_p$, we can identify P to a section of $\mathcal{O}(p)$ and thus defining $P \cdot s = P \otimes s$, we obtain a section of $\mathcal{O}(p) \otimes \mathcal{F}(q) \cong \mathcal{F}(q)(p) \cong \mathcal{F}(p + q)$, giving $\Gamma(\mathcal{F})$ a structure of S-module compatible with the graduation. An equivalent way of defining this multiplication is in terms of components, since $s = (s_i)_{i=0}^r$ with $s_i \in \Gamma(U_i, \mathcal{F})$ and $s_i = (t_j^q/t_i^q)s_j$ over $U_i \cap U_j$, thus defining $(P \cdot s)_i = (P/t_i^p)s_i$ yields the same result. This assignment is again functorial.

Definition 17. *We wish to compare the functors $\mathcal{A}(M)$ and $\Gamma(\mathcal{F})$.*

Let M be a graduated S-module, $m \in M_0$. We have $m/1 \in M_x$ varying continuously with $x \in X$, thus defining a section $\alpha(m) \in \Gamma(X, \mathcal{A}(M))$. If we have $m \in M_n$, then $m \in M(n)_0$ and defines a section $\alpha(m) \in \Gamma(X, \mathcal{A}(M(n))) = \Gamma(X, \mathcal{A}(M)(n))$. This defines a homomorphism $\alpha : M \to \Gamma(\mathcal{A}(M))$.

Let $\mathcal{F}$ be an algebraic sheaf over X, $s/Q \in \Gamma(\mathcal{F})_x$ with $s \in \Gamma(X, \mathcal{F}(n))$, $Q \in S_n$ and $Q(x) \neq 0$. The fraction $1/Q$ is homogeneous of degree $-n$ and regular on x, thus is in $\mathcal{O}(-n)_x$. It follows that $1/Q \otimes s \in \mathcal{O}(-n)_x \otimes \mathcal{F}(n)_x \cong \mathcal{F}_x$, an element denoted $\beta_x(s/Q)$ (as before, we can define componentwise $\beta_x(s/Q) = (t_i^n/Q)s_i(x)$ when $x \in U_i$). The collection β_x with $x \in X$ defines a homomorphism $\beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}$.

An immediate computation yields the following:

Proposition 7. *Let M be a graduated S-module, the composition $\beta \circ \mathcal{A}(\alpha) : \mathcal{A}(M) \to \Gamma(\mathcal{A}(M)) \to \mathcal{A}(M)$ is the identity $\mathrm{id}_{\mathcal{A}(M)}$. Let $\mathcal{F}$ be an algebraic sheaf over X, the* *composition* $\Gamma(\beta)\circ\alpha : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{A}(\Gamma(\mathcal{F}))) \to \Gamma(\mathcal{F})$ *is the identity* $\mathrm{id}_{\Gamma(\mathcal{F})}$.

When $\mathcal{L} = \bigoplus_{j\in J}\mathcal{O}(n_j)$ is an algebraic sheaf over X, J being finite, then $\Gamma(\mathcal{L})$ verifies condition (TF) and $\beta : \mathcal{A}(\Gamma(\mathcal{L})) \to \mathcal{L}$ is a bijection. In fact:

Theorem 9. *Let $\mathcal{F}$ be an algebraic coherent sheaf over X. Then there is a graduated S-module M verifying condition (TF) such that $\mathcal{A}(M) \cong \mathcal{F}$.*

Proof. We shall only sketch it. As a consequence of Theorem 8, there is an exact sequence of algebraic sheaves $\mathcal{L}^1 \stackrel{\varphi}{\to} \mathcal{L}^0 \to \mathcal{F} \to 0$ with $\mathcal{L}^1 = \bigoplus_{j\in J_1}\mathcal{O}(n_j)$ and $\mathcal{L}^0 = \bigoplus_{j\in J_0}\mathcal{O}(n_j)$. For $\Gamma(\varphi) : \Gamma(\mathcal{L}^1) \to \Gamma(\mathcal{L}^0)$ let $M = \mathrm{Coker}(\Gamma(\varphi))$, by the above we know that it verifies condition (TF). Applying the functor $\mathcal{A}$ to the exact sequence $\Gamma(\mathcal{L}^1) \to \Gamma(\mathcal{L}^0) \to M \to 0$, we obtain the exact sequence $\mathcal{A}(\Gamma(\mathcal{L}^1)) \to \mathcal{A}(\Gamma(\mathcal{L}^0)) \to \mathcal{A}(M) \to 0$. Consider the commutative diagram:

$$\begin{array}{ccccccc} \mathcal{A}(\Gamma(\mathcal{L}^1)) & \longrightarrow & \mathcal{A}(\Gamma(\mathcal{L}^0)) & \longrightarrow & \mathcal{A}(M) & \longrightarrow & 0\,, \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle\beta}\downarrow & & & & \\ \mathcal{L}^1 & \longrightarrow & \mathcal{L}^0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

we have that both vertical homomorphisms are bijections, and thus $\mathcal{A}(M) \cong \mathcal{F}$. □

When studying the cohomology, we shall use this equivalence and work over modules. With the usual notations $X = \mathbb{P}_r(\mathbb{K})$, $I = \{0,\dots,r\}$ and $S = \mathbb{K}[t_0,\dots,t_r]$, let M be a graduated S-module and $k, q \in \mathbb{N}$. We define a group $C_k^q(M)$ through its elements, which are maps $m\langle\cdot\rangle : I^{q+1} \to M_{k(q+1)}$ with the element $m\langle i_0 \dots i_q\rangle$ being alternating. Via the obvious internal sum and multiplication by elements of $\mathbb{K}$, $C_k^q(M)$ has a structure of *vector space* over $\mathbb{K}$. Set $C_k(M) = \bigoplus_{q=0}^{r} C_k^q(M)$, it is a complex with the differential $d : C_k^q(M) \to C_k^{q+1}(M)$ given by $(dm)\langle i_0\cdots i_{q+1}\rangle = \sum_{j=0}^{q+1}(-1)^j t_{i_j}^k m\langle i_0\cdots \hat{i_j}\cdots i_{q+1}\rangle$, where $d\circ d = 0$. We denote by $H_k^q(M)$ the q-cohomology group of $C_k(M)$. Given $k \le h \in \mathbb{Z}$, we have a homomorphism $\rho_k^h : C_k^q(M) \to C_h^q(M)$ via $\rho_k^h(m)\langle i_0\cdots i_q\rangle = (t_{i_0}\cdots t_{i_q})^{h-k} m\langle i_0\cdots i_q\rangle$ that commutes with the differential and $\rho_h^l \circ \rho_k^h = \rho_k^l$ when $k \le h \le l$. This allow us to define the complex $C(M) = \varinjlim_{k\to\infty} C_k(M)$, whose cohomology groups are $H^q(M) = \varinjlim_{k\to\infty} H_k^q(M)$ since cohomology commutes with inductive limits. A homomorphism $\varphi : M \to M'$ defines a homomorphism $\varphi : C_k(M) \to C_k(M')$ via $\varphi(m)\langle i_0\cdots i_q\rangle = \varphi(m\langle i_0\cdots i_q\rangle)$, thus taking limits a homomorphism $\varphi : C(M) \to C(M')$. Those two homomorphisms commute with the differential, defining the homomorphisms $\varphi : H_k^q(M) \to H_k^q(M')$ and $\varphi : H^q(M) \to H^q(M')$. As usual, an exact sequence of graduated S-modules $0 \to M \to M' \to M'' \to 0$ gives rise to the exact sequences of complexes $0 \to C_k(M) \to C_k(M') \to C_k(M'') \to 0$ and $0 \to C(M) \to C(M') \to C(M'') \to 0$, thus in cohomology to $\cdots \to H_k^q(M') \to H_k^q(M'') \to H_k^{q+1}(M) \to H_k^{q+1}(M') \to \cdots$ and $\cdots \to H^q(M') \to H^q(M'') \to H^{q+1}(M) \to H^{q+1}(M') \to \cdots$ both long exact sequences.

We are interested in computing $H^q(M)$ in the general case. For this, the natural way to proceed is by induction over the dimension of M as a graduated S-module. It is thus interesting to first see how the cohomology of certain modules, that will be the ones of lesser dimension, behaves. Let $m \in M_0$, we have $\alpha^k(m) = \{t_i^k m\}_{i\in I} \in C_k^0(M)$ a 0-cocycle, by being precisely in degree 0 it can be identified to its cohomology class. This defines a $\mathbb{K}$-linear homomorphism $\alpha^k : M_0 \to H_k^0(M)$ with $\alpha^k = \rho_k^h \circ \alpha^h$ when $h \geq k$, thus taking limits we obtain a homomorphism $\alpha : M_0 \to H^0(M)$. Consider $(P_0, \ldots, P_h) \in S^h$, we denote $(P_0, \ldots, P_h)M = \{\sum_{i=0}^h P_i m_i | m_i \in M\}$ a submodule of M, that is homogeneous if the P_i are homogeneous. For $P \in S$ and N a submodule of M, we denote $N : P = \{m \in M | Pm \in N\}$, that is homogeneous if N and P are homogeneous.

Proposition 8. *Let M be a graduated S-module and $k \in \mathbb{N}$. Suppose that for every $i \in I$ we have $(t_0^k, \ldots, t_{i-1}^k)M : t_i^k = (t_0^k, \ldots, t_{i-1}^k)M$. Then:*

1. *$\alpha^k : M_0 \to H_k^0(M)$ is bijective (if $r \geq 1$),*
2. *$H_k^q(M) = 0$ for $0 < q < r$.*

Proof. It is a particular case of a result in [3]. □

Taking $S(n)$ as graduated S-module above, we obtain:

Proposition 9. *Let $k \in \mathbb{N}$, $n \in \mathbb{Z}$, then:*

1. *$\alpha^k : S_n \to H_k^0(S(n))$ is bijective (if $r \geq 1$),*
2. *$H_k^q(S(n)) = 0$ for $0 < q < r$.*
3. *$H_k^r(S(N))$ admits as a basis over $\mathbb{K}$ the cohomology classes of the monomials $t_0^{\alpha_0} \cdots t_r^{\alpha_r}$ with $0 \leq \alpha_i < k$ and $\sum_{i=0}^r \alpha_i = k(r+1) + n$.*

By a change of language and using how ρ_k^h acts on the monomials above, it follows:

Corollary 4. *For $k \geq -n - r$, $H_k^r(S(n))$ admits as a basis over $\mathbb{K}$ the cohomology classes of the monomials $(t_0 \cdots t_r)^k / t_0^{\beta_0} \cdots t_r^{\beta_r}$ with $\beta_i > 0$ and $\sum_{i=0}^r \beta_i = -n$. For $h \geq k$, the homomorphism $\rho_k^h : H_k^q(S(n)) \to H_h^q(S(n))$ is a bijection for every $q \geq 0$. The homomorphism $\alpha : S_n \to H^0(S(n))$ is bijective if $r \geq 1$ or $n \geq 0$. We have $H^q(S(n)) = 0$ for $0 < q < r$, $H^r(S(n))$ is a vector space over $\mathbb{K}$ of dimension $\binom{-n-1}{r}$.*

We now consider the general case over modules satisfying condition (TF):

Proposition 10. *Let M be a graduated S-module verifying condition (TF). Then:*

1. *There is $k(M) \in \mathbb{Z}$ such that $\rho_k^h : H_k^q(M) \to H_h^q(M)$ is bijective for $h \geq k \geq k(M)$ and every $q \in I$.*
2. *$H^q(M)$ is a vector space over $\mathbb{K}$ of finite dimension for every $q \in I$.*
3. *There is $n(M) \in \mathbb{Z}$ such that for $n \geq n(M)$ we have $\alpha : M_n \to H^0(M(n))$ a bijection and $H^q(M(n)) = 0$ for $q > 0$.*

Proof. We shall only sketch it. Since M verifies condition (TF), we can take M of finite type. As usual, we say that *dimension* of M is $\dim(M) \leq s$ for $s \in \mathbb{N}$ if there is an exact sequence $0 \to L^s \to \cdots \to L \to M \to 0$ of graduated S-modules, the L^i being of finite type for $i = 0, \dots, s$. By Hilbert's syzygy Theorem (see [5]), $\dim(M) \leq r+1$ always.

As foretold, we will use induction over $\dim(M)$. If $\dim(M) = 0$, then M is free of finite type, thus $M = \bigoplus_{j \in J} S(n_j)$ and the result follows from Corollary 4. Let $\dim(M) \leq s$, consider $N = \operatorname{Ker}(L^0 \to M)$ and the exact sequence $0 \to N \to L^0 \to M \to 0$. The result is true for N and L^0 by the induction hypothesis. Consider the commutative diagram:

$$\begin{array}{ccccccccc} H^q_k(N) & \longrightarrow & H^q_k(L^0) & \longrightarrow & H^q_k(M) & \longrightarrow & H^{q+1}_k(N) & \longrightarrow & H^{q+1}_k(L^0)\ , \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^q_h(N) & \longrightarrow & H^q_h(L^0) & \longrightarrow & H^q_h(M) & \longrightarrow & H^{q+1}_h(N) & \longrightarrow & H^{q+1}_h(L^0) \end{array}$$

where $h \geq k \geq \sup(k(N), k(L^0))$, applying the Five Lemma (see [9]) the first point is proved, and since $H^q_k(M)$ is of finite dimension over $\mathbb{K}$ for being in a cohomology sequence with both $H^q_k(L^0)$ and $H^{q+1}_k(N)$ of finite dimension over $\mathbb{K}$, then the second point is also proved.

Consider now $n \geq \sup(n(L^0), n(N))$. The exact sequence $H^q(L^0(n)) \to H^q(M(n)) \to H^{q+1}(N(n))$ shows that $H^q(M(n)) = 0$. Consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_n & \longrightarrow & L^0_n & \longrightarrow & M_n & \longrightarrow & 0\ , \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow \\ 0 & \longrightarrow & H^0(N(n)) & \longrightarrow & H^0(L^0(N)) & \longrightarrow & H^0(M(n)) & \longrightarrow & 0 \end{array}$$

it shows that $\alpha : M_n \to H^0(M(n))$ is a bijection, proving the third point. □

This will enable the comparison of the cohomology groups $H^q(M)$ and $H^q(X, \mathcal{A}(M))$ when M verifies condition (TF). Since $H^q(M)$ is defined via alternating cochains, we will first compare $C(M)$ with $C'(\mathfrak{U}, \mathcal{A}(M))$. Consider $m \in C^q_k(M)$, let $(i_0, \dots, i_q) \in I^{q+1}$, we clearly have $(t_{i_0} \cdots t_{i_q})^k \in S(U_{i_0 \cdots i_q})$, and thus $m\langle i_0 \cdots i_q \rangle / (t_{i_0} \cdot t_{i_q})^k \in M_{U_{i_0 \cdots i_q}}$. The system $\iota_k(m)$ formed by varying $(i_0, \dots, i_q) \in I^{q+1}$ is an alternating q-cochain of $\mathfrak{U}$ with values in $\mathcal{A}(M)$. We immediately see that ι_k commutes with the differential and $\iota_k = \iota_k \circ \rho^h_k$ when $h \geq k$, thus by taking limits they define a homomorphism $\iota : C(M) \to C'(\mathfrak{U}, \mathcal{A}(M))$ that commutes with the differential.

Proposition 11. *Let M be a graduated S-module verifying condition (TF), then $\iota : C(M) \to C'(\mathfrak{U}, \mathcal{A}(M))$ is bijective.*

When $\mathcal{A}(M)$ is coherent, $H'^q(\mathfrak{U}, \mathcal{A}(M)) \cong H^q(\mathfrak{U}, \mathcal{A}(M)) \cong H^q(X, \mathcal{A}(M))$, obtaining:

Corollary 5. *Let M be a graduated S-module verifying condition (TF), then $\iota : C(M) \to C'(\mathfrak{U}, \mathcal{A}(M))$ defines an isomorphism $H^q(M) \cong H^q(X, \mathcal{A}(M))$ for every $q \geq 0$.*

Several results that follow from the obtained above are that whenever M a graduated S-module verifies condition (TF), then the homomorphism $\alpha : M \to \Gamma(\mathcal{A}(M))$ is $\mathscr{C}$-bijective, thus when $\mathcal{F}$ is an algebraic coherent sheaf over X, $\Gamma(\mathcal{F})$ is a graduated S-module that verifies condition (TF) and the homomorphism $\beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}$ is bijective. Moreover, the cohomology groups $H^q(X, \mathcal{F})$ are vector spaces of finite dimension over $\mathbb{K}$ for every $q \geq 0$, and $H^q(X, \mathcal{F}(n)) = 0$ for $q \geq 0$ and $n \in \mathbb{Z}$ big enough. In the special case of $\mathcal{O}$, we have that $H^q(X, \mathcal{O}(n)) = 0$ for $0 < q < r$ and $H^r(X, \mathcal{O}(n))$ is a vector space over $\mathbb{K}$ of dimension $\binom{-n-1}{r}$, having for a basis the cohomology classes of the monomials $1/t_0^{\beta_0} \cdots t_r^{\beta_r}$ with $\beta_i > 0$ and $\sum_{i=0}^{r} \beta_i = -n$. In particular, $H^r(X, \mathcal{O}(-r-1))$ is a vector space over $\mathbb{K}$ of dimension 1 and has the cohomology class of $1/t_0 \cdots t_r$ as a basis.

We can now study the behavior of algebraic coherent sheaves over projective varieties. As a particular case of algebraic varieties, when V is a closed subvariety of the projective space X and $\mathcal{F}$ is an algebraic coherent sheaf over V, by extending by 0 outside of V we obtain $\mathcal{F}^X$ an algebraic coherent sheaf over X satisfying $H^q(X, \mathcal{F}^X) = H^q(V, \mathcal{F})$. The study of algebraic coherent sheaves over projective varieties is then a consequence of the study of algebraic coherent sheaves over the projective space that we just saw, thus by the above:

Theorem 10. *Let V be a projective variety, $\mathcal{F}$ an algebraic coherent sheaf over V, then $H^q(V, \mathcal{F})$ are vector spaces over $\mathbb{K}$ of finite dimension and are trivial for $q > \dim(V)$.*

In particular, $H^0(V, \mathcal{F}) = \Gamma(V, \mathcal{F})$ is a vector space over $\mathbb{K}$ of finite dimension. By considering the covering $\mathfrak{U}' = \{U_i'\}_{i \in I} = \{U_i \cap V\}_{i \in I}$ of V, the analogous construction of $\mathcal{F}(n)$ as in Definition 14 yields a generalization for V, while having the same formal properties, in particular there is a canonical isomorphism $\mathcal{F}(n) \cong \mathcal{F} \otimes \mathcal{O}_V(n)$. Moreover, $\mathcal{F}^X(n) = \mathcal{F}(n)^X$ and thus we obtain the desired result:

Theorem 11. *Let V be a projective variety, $\mathcal{F}$ an algebraic coherent sheaf over V. There is $m(\mathcal{F}) \in \mathbb{Z}$ such that for every $n \geq m(\mathcal{F})$ we have:*

1. *For every $x \in V$, $\mathcal{F}(n)_x$ is generated by $\Gamma(V, \mathcal{F}(n))$ as an $\mathcal{O}_{x,V}$-module.*
2. *$H^q(V, \mathcal{F}(n)) = 0$ for every $q > 0$.*

5 Conclusion

The aim of the reading of the article, which was to determine the cohomology of affine and projective varieties with values over algebraic coherent sheaves, has been attained. Through the study of the identifications that relate finitely generated A-modules in the first case and graduated S-modules verifying condition (TF) in the second case with $\mathcal{F}$ algebraic coherent sheaves, we achieved the expected results:

1. $H^q(V, \mathcal{F}) = 0$ for V an affine variety,

2. $H^q(V, \mathcal{F}) = 0$ for V a projective variety when $q > \dim(V)$,

and restricting us to the case of V a projective variety, the not so obvious ones:

3. $H^q(V, \mathcal{F})$ is a finite dimensional $\mathbb{K}$-vector space,

4. $H^q(V, \mathcal{F}(n)) = 0$ for $n \in \mathbb{Z}$ big enough.

The results presented in this dissertation cover most of J.-P. Serre's article, although not all of it has been featured: we stopped at the computation above. There is a way of identifying the functors $\mathrm{Ext}^q_{\mathcal{O}_x}(\mathcal{O}_X, \mathcal{F})$ with the cohomology groups $H^q(X, \mathcal{F})$ that sometimes allows a more elegant and instructive ways of considering several of the results, for example Corollary 2.

This is a natural continuation for further study and expansion that strengthens the modern approach to the subject.

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