Triality: a particularity of Spin(8)

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Introduction

A duality is a well know concept:

Definition

Let V_1, V_2 be two vector spaces over a field $\mathbb{K}$. We say that there is a **duality** *between them if there exists a linear map:*

$$f : V_1 \otimes V_2 \longrightarrow \mathbb{K}.$$

That is non degenerate:

$$f(v_1 \otimes v_2) = 0 \text{ for every } v_i \iff v_j = 0.$$

We wish to understand **triality**, an analogous construction over three vector spaces. We will use Clifford structures to define $\mathrm{Spin}(8)$ and use his representations as the three spaces.

The Classical groups and Representations

This work has a strong basis on some Classical groups and Representations:

Definition

The **general linear group** *and the (real)* **orthogonal group** *are:*

$$\mathrm{GL}_n(\mathbb{K}) = \{A \in \mathrm{M}_n(\mathbb{K}) : \det A \neq 0\},\ \mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = \mathrm{Id}_n\}.$$

Definition

Let $(\mathbb{V}, |\ \ |)$ *be a finite dimensional normed* $\mathbb{K}$ *vector space,* G *be a matrix group that has a continuous homomorphism* $\varphi : G \to \mathrm{GL}_{\mathbb{K}}(\mathbb{V})$*. The associated action:*

$$\begin{array}{rccc} \mu_\varphi & : & G \times \mathbb{V} & \longrightarrow & \mathbb{V} \\ & & (g, v) & \longmapsto & \varphi(g)(v) \end{array}$$

is called a **(continuous) linear action** *or* **representation** *of* G *on* $\mathbb{V}$*.*

Clifford Algebras: definition and structure

To construct $\mathrm{Spin}(8)$, we will use several Clifford objects:

Definition

We define the **real Clifford algebra** *in* $n \in \mathbb{N}$ *variables* Cl_n *as the* $\mathbb{R}$ *algebra generated by the elements* $e_1, \ldots, e_n \in \mathrm{Cl}_n$ *for which:*

$$\begin{cases} e_s e_r = -e_r e_s \text{ if } s \neq r, \\ e_r^2 = -1. \end{cases}$$

Properties

There is a **canonical automorphism** $\alpha : \mathrm{Cl}_n \to \mathrm{Cl}_n$ and a **conjugation** $\overline{(\ )} : \mathrm{Cl}_n \to \mathrm{Cl}_n$ determining a $\pm$-**grading** and a **norm**.

Theorem (Bott periodicity)

For $n \in \mathbb{N}$; $\mathrm{Cl}_{n+8} \cong \mathrm{Cl}_n \otimes \mathrm{M}_{16}(\mathbb{R})$ and $\mathrm{Cl}_{n+2} \otimes \mathbb{C} \cong (\mathrm{Cl}_n \otimes \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{M}_2(\mathbb{C})$.

Clifford groups

Definition

Given $n \geq 1$, we define the **Clifford group** Γ_n *as the subgroup:*

$$\Gamma_n = \{u \in \mathrm{Cl}_n^\times : \alpha(u)xu^{-1} \in \mathbb{R}^n \text{ for all } x \in \mathbb{R}^n\}.$$

Proposition

There is a continuous group homomorphism:

$$\begin{array}{cccc} \rho & : & \Gamma_n & \longrightarrow & \mathrm{O}(n) \\ & & u & \longmapsto & \rho_u \end{array} \quad \text{with} \quad \begin{array}{cccc} \rho_u & : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & v & \longmapsto & \alpha(u)vu^{-1} \end{array} \text{ an isometry.}$$

Proposition

There is a continuous group homomorphism:

$$\begin{array}{cccc} \nu & : & \Gamma_n & \longrightarrow & \mathbb{R}^\times \\ & & u & \longmapsto & u\overline{u} \end{array}$$

$\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$: definition and characterization

Definition

Given $n \geq 1$, *we define the* **pinor group** $\mathrm{Pin}(n)$ *and the* **spinor group** $\mathrm{Spin}(n)$ *as:* $\mathrm{Pin}(n) = \ker \nu$, $\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cl}_n^+$.

In fact, $\mathrm{Pin}(n)$ to $\mathbb{S}^{n-1}$ and $\mathrm{Spin}(n)$ to $\mathrm{SO}(n)$ are intimately related:

Theorem

- It holds $\langle \mathbb{S}^{n-1} \rangle = \mathrm{Pin}(n)$, where $\mathbb{S}^{n-1} = \left\{ \sum_{r=1}^n x_r e_r : \sum_{r=1}^n x_r^2 = 1 \right\}$.
- The map $\rho^+ : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is surjective with $\ker \rho^+ = \{\pm 1\}$.

Examples

Spin group	Classical group	Spin group	Classical group
$\mathrm{Spin}(1)$	$\mathrm{O}(1)$	$\mathrm{Spin}(4)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
$\mathrm{Spin}(2)$	$\mathrm{SO}(2) \cong \mathrm{U}(1)$	$\mathrm{Spin}(5)$	$\mathrm{Sp}(2)$
$\mathrm{Spin}(3)$	$\mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong \mathbb{S}^3$	$\mathrm{Spin}(6)$	$\mathrm{SU}(4)$

The $\mathrm{Spin}(n)$ representations

$\mathrm{Spin}(n)$ always has a representation $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n) \subset \mathrm{GL}_n(\mathbb{C})$.

Proposition

Let $n \in \mathbb{N}$. If $n = 2r + 1$ is odd, then $\mathrm{Spin}(n)$ has one irreducible representation Δ of degree 2^r. If $n = 2r$ is even, then $\mathrm{Spin}(n)$ has two irreducible representations Δ^+, Δ^- of degree 2^{r-1}.

Proposition

Let $r \in \mathbb{N}$. The representation Δ of $\mathrm{Spin}(2r+1)$ is real if $2r + 1 \equiv 1, 7 \mod 8$. The representations Δ^+ and Δ^- of $\mathrm{Spin}(2r)$ are real if $2r \equiv 0 \mod 8$.

Clearly $\mathrm{Spin}(8)$ must be special, since $\dim(\lambda) = \dim(\Delta^+) = \dim(\Delta^-) = 8$.

Theorem

The only irreducible representations of $\mathrm{Spin}(8)$ are λ, Δ^+ and Δ^-.

Automorphisms: definitions and characterization

Definition

Let A be an algebra over $\mathbb{K}$.

- *The group of K-**algebra automorphisms** is $\mathrm{Aut}_{\mathbb{K}}(A)$.*
- *Any conjugation by a unit element $u \in A^{\times}$ is called an **inner automorphism** forming the group of inner automorphisms $\mathrm{Inn}_{\mathbb{K}}(A)$.*
- *The **group of outer automorphisms** is defined as $\mathrm{Out}_{\mathbb{K}}(A) = \mathrm{Aut}_{\mathbb{K}}(A)/\mathrm{Inn}_{\mathbb{K}}(A)$. The equivalence classes of $\mathrm{Out}_{\mathbb{K}}(A)$ are called **outer automorphisms**.*

The elements of $\mathrm{Out}_{\mathbb{R}}(\mathrm{Spin}(8))$ are in a one to one correspondence with the permutations of the three representations λ, Δ^+ and Δ^-:

Theorem

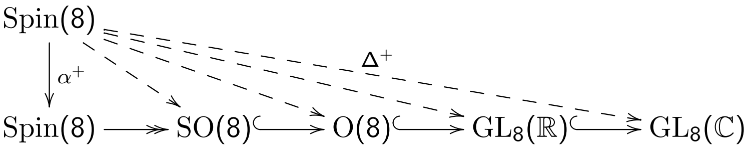
It holds $\mathrm{Out}_{\mathbb{K}}(\mathrm{Spin}(8)) = \Sigma_3\{\lambda, \Delta^+, \Delta^-\}$.

Sketch of the proof: the morphism is surjective

Define the homomorphism:

$$\begin{array}{rcl} \psi : \mathrm{Out}_{\mathbb{R}}(\mathrm{Spin}(8)) & \longrightarrow & \Sigma_3\{\lambda, \Delta^+, \Delta^-\} \\ \alpha & \longmapsto & \psi(\alpha) \end{array} \quad : \quad \begin{array}{rcl} \{\lambda, \Delta^+, \Delta^-\} & \longrightarrow & \{\lambda, \Delta^+, \Delta^-\} \\ [\rho] & \longmapsto & [\rho \circ \alpha] \end{array}$$

Consider the diagram:

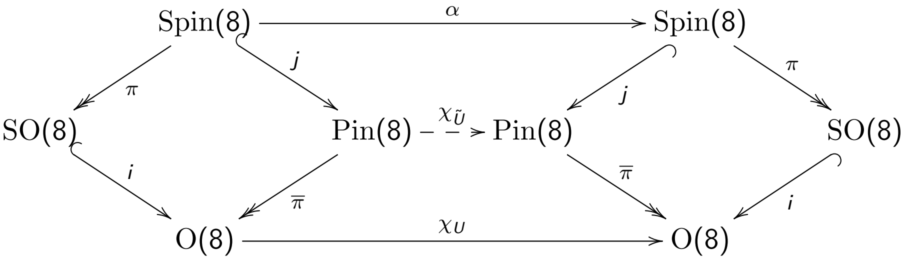


Which yields two permutations $\psi(\alpha^+)$ and $\psi(\alpha^-)$ that generate Σ_3:

$\psi(\alpha^+)$	$\psi(\alpha^-)$	Result
2 or 3	3	Two permutations of different order generate Σ_3
3	2	Two permutations of different order generate Σ_3
2	2	Two different permutations of order 2 generate Σ_3

Sketch of the proof: the morphism is injective

Let $\alpha : \mathrm{Spin}(8) \to \mathrm{Spin}(8)$ be an automorphism with $\psi(\alpha) = \mathrm{Id}_{\Sigma_3}$. We showed that it is an inner automorphism. Consider the diagram:



Which commutes and in fact $\alpha = \chi_{\tilde{U}}$, it is a conjugation by an element $\tilde{U} \in \mathrm{Spin}(8)$, as if $\tilde{U} \notin \mathrm{Spin}(8)$ then $\alpha = \chi_{e_1}$ with a contradiction:

$$\Delta^+ \neq \Delta^+ \circ \alpha = \psi(\alpha)(\Delta^+) = \Delta^+.$$

Triality: definition

Definition

Let V_1, V_2, V_3 *be three finite dimensional vector spaces over a field* $\mathbb{K}$*. We say that there is a* **triality** *between them if there exists a linear map:*

$$f : V_1 \otimes V_2 \otimes V_3 \longrightarrow \mathbb{K}.$$

That is non degenerate:

$$f(v_1 \otimes v_2 \otimes v_3) = 0 \text{ for every } v_i \iff v_j = 0 \text{ or } v_k = 0.$$

Examples

Consider $K = \mathbb{R}$ and $V = V_1 = V_2 = V_3 = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we have a triality:

$$\begin{array}{cccc} f & : & V \otimes V \otimes V & \longrightarrow & \mathbb{R} \\ & & x \otimes y \otimes z & \longmapsto & \Re(xyz) \end{array}$$

Having a triality is not as easy as having a duality.

Triality and division algebras

Definition

Let $f : V_1 \otimes V_2 \otimes V_3 \longrightarrow \mathbb{R}$ be a triality over three finite dimensional real vector spaces and $\tilde{f}$, φ, ψ as above. We define $\Phi = \tilde{f} \circ (\psi \otimes \varphi)^{-1}$.

Consider $v_1 \in V_1$, $v_2 \in V_2$, $0 \neq e_1 \in V_1$ and $0 \neq e_2 \in V_2$, define:

$$\begin{array}{rcl} \tilde{f}(v_1 \otimes v_2) : V_3 & \longrightarrow & \mathbb{R} \\ v_3 & \longmapsto & f(v_1 \otimes v_2 \otimes v_3) \end{array} \quad \begin{array}{rcl} \varphi : V_2 & \longrightarrow & V_3^* \\ v_2 & \longmapsto & \tilde{f}(e_1 \otimes v_2) \end{array} \quad \begin{array}{rcl} \psi : V_1 & \longrightarrow & V_3^* \\ v_1 & \longmapsto & \tilde{f}(v_1 \otimes e_2) \end{array}$$

Proposition

The map $\Phi : V_3^* \otimes V_3^* \rightarrow V_3^*$ has an identity element and no zero divisors.

Φ is a “product” in V_3^*, providing it with a structure of $\mathbb{R}$ division algebra.

Theorem

A finite dimensional real division algebra has dimension 1, 2, 4 or 8 ($\mathbb{O}$).

The $\mathrm{Spin}(8)$ representations induce a triality

Proposition

For the irreducible representations of $\mathrm{Spin}(8)$ we have that:

- The representation λ is self dual, that is, $\lambda^* = \lambda$.
- The equality $\Delta^+ \otimes \Delta^- = \lambda + \Theta$, where Θ is some representation.
- The group $\mathrm{Spin}(8)$ acts transitively over $\mathbb{S}^7 \times \mathbb{S}^7 \subset \lambda \times \Delta^+$.

Note that we interpreted the representations as the vector spaces the group acts on, with $\lambda \otimes \Delta^+ \otimes \Delta^- = \lambda \otimes (\lambda + \Theta) = \lambda \otimes \lambda + \lambda \otimes \Theta$.

Definition

Given $\lambda \otimes \Delta^+ \otimes \Delta^- \ni v = w = (w_1 \otimes w_2) + (w_3 \otimes w_4) \in \lambda \otimes \lambda + \lambda \otimes \Theta$, *define* $f : \lambda \otimes \Delta^+ \otimes \Delta^- \to \mathbb{R}$ *as* $f(v) = \mu(w_1 \otimes w_2)$, $\mu : \lambda \otimes \lambda^* \to \mathbb{R}$.

Theorem

The map $f : \lambda \otimes \Delta^+ \otimes \Delta^- \to \mathbb{R}$ is a triality.

Triality on $\mathrm{Spin}(8)$

Sketch of the proof:

If any element $x, y, z = 0$, then $f(x \otimes y \otimes z) = 0$. Suppose there exist $x, y \in \mathbb{S}^7 \subset \mathbb{R}^8$ with $\tilde{f}(x \otimes y) = 0$. There exist $x_0, y_0 \in \mathbb{S}^7 \subset \mathbb{R}^8$ for which $\tilde{f}(x_0 \otimes y_0) \neq 0$ and $g \in \mathrm{Spin}(8)$ with $g \cdot x = x_0$ and $g \cdot y = y_0$. Thus:

$$0 = \tilde{f}(x \otimes y) = g \cdot \tilde{f}(x \otimes y) = \tilde{f}(g \cdot x \otimes g \cdot y) = \tilde{f}(x_0 \otimes y_0) \neq 0, \text{ contradiction.}$$

Similarly, the following yields a contradiction:

$$\begin{array}{rccccccc}
f \circ \alpha_{(2,3)} & : & \lambda \otimes \Delta^+ \otimes \Delta^- & \overset{(2,3)}{\longleftrightarrow} & \lambda \otimes \Delta^- \otimes \Delta^+ & \longrightarrow & \mathbb{R} \\
 & & x \otimes \quad \otimes z & \longmapsto & x \otimes \quad \otimes z & \longmapsto & f(x \otimes \quad \otimes z), \\
f \circ \alpha_{(1,2,3)} & : & \lambda \otimes \Delta^+ \otimes \Delta^- & \overset{(1,2,3)}{\longleftrightarrow} & \Delta^- \otimes \lambda \otimes \Delta^+ & \longrightarrow & \mathbb{R} \\
 & & \quad \otimes y \otimes z & \longmapsto & \quad \otimes y \otimes z & \longmapsto & f(\quad \otimes y \otimes z).
\end{array}$$

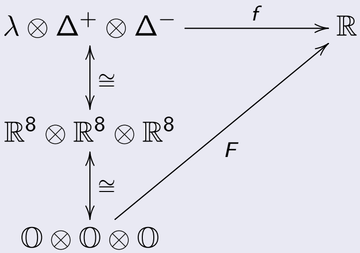
Theorem

We can identify each and every one of λ, Δ^+, Δ^- with $\mathbb{O}$.

- Beginning with the Clifford Algebras, we defined the Clifford group Γ_n, the pinor group $\mathrm{Pin}(n)$ and the spinor group $\mathrm{Spin}(n)$.
- We showed that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$ and characterized the outer automorphisms of $\mathrm{Spin}(8)$: $\mathrm{Out}_{\mathbb{K}}(\mathrm{Spin}(8)) = \Sigma_3$.
- We constructed a **triality** over the representations of $\mathrm{Spin}(8)$.

Observation

In fact, we have the commutative diagram:



with $F(x \otimes y \otimes z) = \Re(xyz)$

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