B_∞ STRUCTURES, MONOIDAL CATEGORIES, AND SINGULARITY CATEGORIES

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ABSTRACT. This paper is the transcription by the second author of the three-lecture mini-course on B_∞ structures, monoidal categories, and singularity categories given by the first author during the seventeenth edition of the Séminaire Itinérant Géométrie et Physique held at the University of British Columbia in May 2022.

1. Introduction to B_∞ structures: From Hochschild to Getzler-Jones

Throughout this paper we set k a field, A a unital associative (not necessarily commutative) k-algebra, and undecorated tensor products $\otimes$ will be over k. We will denote the *enveloping algebra* of A by $A^e = A \otimes A^{op}$. The *Hochschild cochain complex* of A, denoted by $C(A, A)$, is given by

$$A \to \operatorname{Hom}_k(A, A) \to \operatorname{Hom}_k(A \otimes A, A) \to \cdots \to \operatorname{Hom}_k(A^{\otimes p}, A) \to \cdots$$

with differential

$$\begin{aligned} d(f)(a_0 \otimes \cdots \otimes a_p) &= a_0 f(a_1 \otimes \cdots \otimes a_p) \\ &\quad + \sum_{i=0}^{p-1} (-1)^{i-1} f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p) \\ &\quad + (-1)^p f(a_0 \otimes \cdots \otimes a_{p-1}) a_p. \end{aligned}$$

The *Hochschild cohomology* of A, denoted by $HH^*(A)$, is the homology of its Hochschild cochain complex, namely $HH^*(A) = H^*C(A, A)$. These ideas were pionered by Hochschild [24]. Note that the first differential is given by

$$\begin{aligned} A &\longrightarrow \operatorname{Hom}_k(A, A) \\ a &\longmapsto (b \mapsto ab - ba) \end{aligned}$$

and the second differential by

$$\begin{aligned} \operatorname{Hom}_k(A, A) &\longrightarrow \operatorname{Hom}_k(A \otimes A, A) \\ D &\longmapsto (a \otimes b \mapsto (Da)b - D(ab) + aD(b)) \end{aligned}$$

whence Hochschild cohomology encodes infinitesimal information about the algebra. Namely

$$HH^0(A) = \{a \in A | ab = ba \text{ for all } b \in A\} = Z(A)$$
$$HH^1(A) = \operatorname{Der}_k(A)/\operatorname{InnDer}_k(A) = \operatorname{OutDer}_k(A)$$

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where we recover a commutative algebra in $Z(A)$ the center of A and a Lie algebra in $\mathrm{OutDer}_k(A)$ the outer derivations of A over the field k.

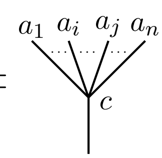
Hochschild cohomology was interpreted by Cartan and Eilenberg [6] and as the k-algebra $HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$ with multiplication given by the cup product by seeing A as the unit object in the category of A^e-modules (equivalently A-bimodules). In fact, Gerstenhaber [16, 17, 18, 19, 20] showed that this cup product makes $HH^*(A)$ into a graded commutative k-algebra, and that $HH^{*-1}(A)$ is a graded Lie algebra with a bracket that controls the deformations of A. This bracket is known as the *Gerstenhaber bracket*, and the structure $(HH^*(A), ?\cup ?, [?, ?])$ arising from the compatibility of this bracket with the cup product is known as *Gerstenhaber algebra*. A modern argument of this fact uses that A is the monoidal unit in the category $D(A^e)$, circumventing the tedious computations of the original proof (see Schwede [50] and Hermann [23]).

This Gerstenhaber bracket, by definition, can be seen as an operation on the Hochschid cochain complex $C(A, A)$. Using brace operations defined by Kadeishvili [30] this interpretation enabled Getzler and Jones [21] to lift the Gerstenhaber algebra structure to a B_∞ algebra structure $(C(A, A), ?\cup ?, \{?, ?\})$.

Remark 1.1.

(1) The nomenclature B_∞ comes from the work of Baues [2] on the monoidal category $D(Sh(X, Ab))$ of sheaves of abelian groups, in particular he proved that the category of singular cochains $C^*_{sg}(X, \mathbb{Z})$ is a B_∞-algebra.

(2) Brace operations come from an operadic viewpoint of operations on cochains, and hence can be visualized as sums of n-ary trees.

$$c\{a_1, \ldots, a_n\} = \sum_{a_1, \ldots, a_n} \pm$$


(3) The B_∞ structure on Hochschild cochains contains all the information of the Gerstenhaber algebra structure on Hochschild cohomology, in particular we can recover the Gerstenhaber bracket as

$$[c, u] = c\{u\} \mp u\{c\}.$$

(4) The B_∞ structure plays a fundamental role in (almost) all the proofs of Deligne's conjecture, stating that the Hochschild cochain complex of an associative ring has a natural action by the singular chains of the little 2-cubes operad. For example, it features prominently in the proofs of McClure and Smith [46], Kontsevich and Soibelman [38], Tamarkin [51], and Lurie [44, Section 5.3].

Formally, a B_∞-*algebra* is a $\mathbb{Z}$-graded k-vector space V together with a dg bialgebra structure on the *bar coalgebra* $\mathbf{B}^+(V)$ of V

$$\mathbf{B}^+(V) = T^c(V) = \bigoplus_{n \in \mathbb{N}} (\Sigma V)^{\otimes n}$$

where ΣV is the suspension of V. In particular, this structure provides a coproduct $\Delta : \mathbf{B}^+(V) \to \mathbf{B}^+(V) \otimes \mathbf{B}^+(V)$ given by

$$\Delta((\Sigma a_1) \otimes \cdots \otimes (\Sigma a_n)) = \sum_{i=0}^{n} ((\Sigma a_1) \otimes \cdots \otimes (\Sigma a_i)) \otimes ((\Sigma a_{i+1}) \otimes \cdots \otimes (\Sigma a_n))$$

for all $a_1, \ldots, a_n \in V$, as well as a coaugmentation $\eta : k \to \mathbf{B}^+(V)$.

Remark 1.2.

(1) The bar coalgebra $\mathbf{B}^+(V)$ is augmented by definition, but V may not be augmented.
(2) The differential $d : \mathbf{B}^+(V) \to \mathbf{B}^+(V)$ yields an A_∞-algebra structure on V. For the rest of the paper we often assume that V is *homologically unital*, namely $H^*(V)$ is unital.
(3) The B_∞-operad is a dg operad whose underlying A_∞ structure can be described by multilinear maps $\mu_l : \mathbf{B}^+(V)^{\otimes l} \to \mathbf{B}^+(V)$ for $l \geq 2$, and whose multiplication can be described by multilinear maps $m_{i,j} : \mathbf{B}^+(V)^{\otimes i} \otimes \mathbf{B}^+(V)^{\otimes j} \to \mathbf{B}^+(V)$ for $i, j \geq 0$. The *braces operad Br* is given by taking the quotient with the operadic ideal generated by the $m_{i,j}$ for $i \geq 2$ and $j \geq 0$. It acts on the Hochschild cochain complex $C(A, A)$ of any A_∞-algebra A, and it is quasi-isomorphic to the E_2-operad when k has characteristic zero (see Kontsevich and Soibelman [38], Willwacher [57, Section 3], and Dolgushev and Willwacher [8]).

Since the Hochschild cochain complex $C(A, A)$ carries the structure of an A_∞-algebra, the augmented bar construction $\mathbf{B}^+(C(A, A))$ inherits a dg bialgebra structure making $C(A, A)$ into a B_∞-algebra.

2. Functoriality of the B_∞ structure on Hochschild cochains

Let A and B be k-algebras. Given $f : A \to B$ a k-algebra morphism, it usually does not induce a morphism $Zf : Z(A) \to Z(B)$ between the centers, and thus it cannot induce a morphism $HH^*f : HH^*(A) \to HH^*(B)$ in Hochschild cohomology. However, we can gain some functoriality by interpreting this over module categories. Let $\mathrm{Mod}A$ be the category of right A-modules and let $\mathrm{End}(\mathrm{id}_{\mathrm{Mod}A})$ be the endomorphism algebra of the identity functor $\mathrm{id}_{\mathrm{Mod}A} : \mathrm{Mod}A \to \mathrm{Mod}A$. Defining the *center* of $\mathrm{Mod}A$ as $Z(\mathrm{Mod}A) = \mathrm{End}(\mathrm{id}_{\mathrm{Mod}A})$, we have a canonical isomorphism

$$\begin{aligned} Z(\mathrm{Mod}A) &\xrightarrow{\sim} Z(A) \\ \varphi &\longmapsto \varphi_A \end{aligned}$$

where we have identified $\varphi_A : A \to A$ with the element $\varphi_A(1_A)$. Given a fully faithful functor $F : \mathrm{Mod}A \to \mathrm{Mod}B$ we then get a restriction morphism

$$\begin{array}{ccc} Z(\mathrm{Mod}B) & \xrightarrow{F^*} & Z(\mathrm{Mod}A) \\ \downarrow \wr & & \downarrow \wr \\ Z(B) & \overset{F^*}{\dashrightarrow} & Z(A) \end{array}$$

where we use that $\mathrm{End}_{\mathrm{Mod}A}(L) \cong \mathrm{End}_{\mathrm{Mod}B}(FL)$ to set $F^*((\varphi_M)_{M \in \mathrm{Mod}B}) = (\psi_L)_{L \in \mathrm{Mod}A}$ as given by considering $\psi_L \in \mathrm{End}_{\mathrm{Mod}A}(L)$ and identifying it with $\varphi_{FL} \in \mathrm{End}_{\mathrm{Mod}B}(FL)$. Our goal is to construct a derived analogue of $F^* : Z(B) \to Z(A)$ by lifting it to Hochschild cochain complexes together with their B_∞ structures.

Consider first $D(A)$ the unbounded derived category of right A-modules. Its objects are complexes of right A-modules, and its morphisms are chain maps between complexes of right A-modules with the particularity that quasi-isomorphisms have formal inverses. Namely given two complexes of right A-modules L and M, if a chain map $s : L \to M$ induces an isomorphism $H^*s : H^*L \to H^*M$ then it has a formal inverse in $D(A)$.

Theorem 2.1 (Keller [35])**.** *Let $X \in D(A)$ be such that the functor $?\otimes^{\mathbb{L}}_A X : D(A) \to D(B)$ is fully faithful. Then there is a canonical restriction morphism*

$$\mathrm{res}_X : C(B, B) \to C(A, A)$$

in the homotopy category of B_∞ algebras. It is invertible if the functor $X\otimes^{\mathbb{L}}_B ? : D(B^{op}) \to D(A^{op})$ is fully faithful.

As a corollary, when A is a Koszul algebra we obtain an isomorphism of B_∞ structures on Hochschild cochain complexes that generalizes the isomorphism of the graded commutative algebra structure on Hochschild cohomologies found by Buchweitz, Green, Snashall, and Solberg [5].

Corollary 2.2 (Keller [35])**.** *Let A be an Adams-graded Koszul algebra and let $A^! = \bigoplus_{p,q} \mathrm{Ext}^p_A(A_0, A_0\langle q\rangle)$ be its Adams-graded Koszul dual viewed as a dg algebra with differential zero. Then we have a canonical isomorphism*

$$C(A, A) \xrightarrow{\sim} C(A^!, A^!)$$

in the homotopy category of Adams-graded B_∞-algebras that induces an isomorphism

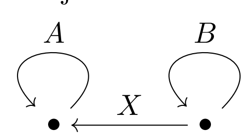
$$HH^*(A) \xrightarrow{\sim} HH^*(A^!)$$

compatible with the cup product and the Gerstenhaber bracket.

The idea behind the proof of Corollary 2.2 is to use the Koszul complex $X = \bigoplus_q A_0\langle q\rangle$ in the unbounded derived category $D^{Adams}(A \otimes (A^!)^{op})$. The proof of Theorem 2.1, which we now sketch, relies on the aforementioned restriction functor F^* and on the generalization of a homotopy bicartesian square to dg categories.

Consider the dg category $\mathcal{G}$ with two objects and three morphisms

$\mathcal{G}$:



where A and B are the given dg algebras and X is an $A \otimes B^e$-module, to which we can associate $C(\mathcal{G}, \mathcal{G})$ the product total complex of the Hochschild cochain complexes of elements in $\mathcal{G}$. Abusing notation, we denote $C(\mathcal{G}, \mathcal{G}) = \mathrm{Hom}^\bullet_k(\mathcal{G}^{\otimes p}, \mathcal{G})$. Note that

$$k[\mathcal{G}] = \left\{ \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \middle| a \in A, b \in B, x \in X \right\}$$

is a dg algebra of upper triangular matrices, having a dg subalgebra of diagonal matrices

$$R = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \subseteq k[\mathcal{G}].$$

With the aforementioned abuse of notation, the cochain complex relative to R is given by $C_R(\mathcal{G}, \mathcal{G}) = \mathrm{Hom}^\bullet_{R^e}(\mathcal{G}^{\otimes p}, \mathcal{G})$. We can see it as a subcomplex $C_R(\mathcal{G}, \mathcal{G}) \subseteq C(\mathcal{G}, \mathcal{G})$ via the inclusions $\mathrm{Hom}_{R^e}(\mathcal{G}^{\otimes p}, \mathcal{G}) \subseteq \mathrm{Hom}_k(\mathcal{G}^{\otimes p}, \mathcal{G})$. Moreover, we can interpret $C_R(\mathcal{G}, \mathcal{G})$ as the Hochschild cochain complex of the dg category $\mathcal{G}$. The inclusion $C_R(\mathcal{G}, \mathcal{G}) \rightarrowtail C(\mathcal{G}, \mathcal{G})$ is a quasi-isomorphism of B_∞-algebras, and we can see $C_R(\mathcal{G}, \mathcal{G})$ as intermediate between $C(A, A)$ and $C(B, B)$. Namely, we want to define $\mathrm{res}_X : C(B, B) \to C(A, A)$ via the following diagram of B_∞-algebra morphisms

$$\begin{array}{ccc} C_R(\mathcal{G},\mathcal{G}) & \xrightarrow{\mathrm{res}_A} & C(A,A) \\ {\scriptstyle \mathrm{res}_B}\downarrow\wr & \nearrow_{\mathrm{res}_X} & \\ C(B,B) & & \end{array}$$

where crucially the restriction $\mathrm{res}_B : C_R(\mathcal{G}, \mathcal{G}) \to C(B, B)$ is a quasi-isomorphism. The reason is that the faithfulness of the functor $? \otimes^{\mathbb{L}}_A X : D(A) \to D(B)$ induces a quasi-isomorphism $A \xrightarrow{\sim} R\mathrm{Hom}_B(X, X)$, completing the diagram

$$\begin{array}{ccccc} C_R(\mathcal{G},\mathcal{G}) & \xrightarrow{\mathrm{res}_A} & C(A,A) & \xleftarrow{\sim} & R\mathrm{Hom}_{A^e}(A,A) \\ {\scriptstyle \mathrm{res}_B}\downarrow\wr & \square_h & \downarrow\wr & & \downarrow\wr \\ C(B,B) & \longrightarrow & R\mathrm{Hom}_{A^{op}\otimes B}(X,X) & \xleftarrow{\sim} & R\mathrm{Hom}_{A^e}(A, R\mathrm{Hom}_B(X,X)) \end{array}$$

where the square on the left is homotopy bicartesian. Defining $\mathrm{res}_X = \mathrm{res}_A \mathrm{res}_B^{-1}$ finishes the proof.

3. B_∞ algebras and monoidal categories

We now follow Lowen and Van den Bergh [43] and Lurie [44, Section 7.1.2] to showcase how the endomorphisms $R\mathrm{End}_{\mathcal{A}}(I)$ of the tensor unit I of a monoidal category $\mathcal{A}$ carry a B_∞ structure that induces several monoidal equivalences of categories.

Given V a homologically unital B_∞-algebra, we denote by $\mathrm{Mod}V$ the category of homologically unital A_∞-modules over V, and by $D(V)$ the associated derived category. Our guide will be the remarkable thesis of Lefèvre-Hasegawa [41].

Lemma 3.1. *The category $D(V)$ has a monoidal triangulated structure with V as the unit.*

Proof. Let $V^+ = V \oplus k$ be the augmented A_∞-algebra of V, let $C^+ = \mathbf{B}^+(V)$, and let $\mathrm{Com}(C^+)$ be the category of cocomplete right dg C^+-comodules (which in this case coincides with the the category of conilpotent right dg C^+-comodules). Since C^+ is a dg bialgebra, $\mathrm{Com}(C^+)$ inherits a monoidal structure via $\otimes$ with k as the unit. We then have

$$\begin{array}{ccccccc} D(V^+) & \longleftarrow & \mathrm{Mod}V^+ & \longleftarrow\!\!\prec & \mathrm{Mod}V & \longrightarrow & D(V) \\ \uparrow\wr\downarrow & & {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & & {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & & \uparrow\wr\downarrow \\ D^{co}(C^+) & \longleftarrow & \mathrm{Com}(C^+) & \longleftarrow\!\!\prec & (\mathrm{Com}(C^+))_{ac} & \longrightarrow & (\mathrm{Com}(C^+))_{ac}[(Rqis)^{-1}] \end{array}$$

where $R = ? \otimes_\tau C^+$ and $L = ? \otimes_\tau V^+$ for $\tau : C^+ \to \Sigma V \cong V \to V$ the canonical twisting cochain, and $D^{co}(C^+)$ is the coderived category, $(\mathrm{Com}(C^+))_{ac}$ is a tensor ideal in $\mathrm{Com}(C^+)$, and $(\mathrm{Com}(C^+))_{ac}[(Rqis)^{-1}]$ is monoidal with unit RV. This induces a monoidal structure on $D(V)$ with unit V via the rightmost vertical equivalence. $\square$

Remark 3.2. It follows that per(V) the perfect derived category of V, which here coincides with thick(V) the thick subcategory generated by V (namely the subcategory of $D(V)$ containing V and being closed under taking shifts, extensions, and retracts), is also monoidal with unit V. In particular, per(V) is a *unitally generated* monoidal triangulated category.

Our philosophy following Remark 3.2 is that every unitally generated monoidal triangulated category should be of this form. Even better, every E_1-monoidal, stable, k-linear ∞-category should be of this form!

Theorem 3.3 (Lowen and Van den Bergh [43])**.** *Let $(\mathcal{A}, \otimes, I)$ be a monoidal k-linear category such that*

(1) $\mathcal{A}$ is abelian (with $\otimes$ not necessarily exact),
(2) $\mathcal{A}$ has enough projectives and $? \otimes P : \mathcal{A} \to \mathcal{A}$ is exact for every projective P.

Then $V = R\mathrm{End}_{\mathcal{A}}(I)$ carries a B_∞ structure such that the canonical equivalence

$$\begin{aligned} \mathrm{per}(V) &\xrightarrow{\sim} \mathrm{thick}(I) \\ V &\longmapsto I \end{aligned}$$

becomes a monoidal equivalence.

Example 3.4. *Let A be a k-algebra, we can identify $\mathcal{A} = \mathrm{Mod}A^e$ with the category of A-bimodules and endow it with a monoidal structure given by $?\otimes_A ? : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and having unit $I = A$. Then $V = R\mathrm{Hom}_{A^e}(A, A) = C(A, A)$ as a dg algebra (up to quasi-isomorphism), and the B_∞ structure given by Theorem 3.3 coincides with the classical B_∞ structure discussed at the end of Section 1.*

Example 3.5. *Let X be a topological space and let $\mathcal{A} = Sh(X, \mathrm{Mod}k)$ with unit $I = \underline{k}_X$. Then $R\mathrm{End}(I) = C^*_{sg}(X, k)$ has a B_∞ structure (see Baues [2]). However, since $\mathcal{A}$ does not have enough projectives, this structure does not come from Theorem 3.3 because it does not apply.*

Let R be an E_2-ring spectrum. Its associated ∞-enhanced derived category $D_\infty(R)$ underlies the E_1-monoidal ∞-stable category $D_\infty(R)^\otimes$, which is compactly generated by its tensor unit R. Let $\mathrm{per}_\infty(R)^\otimes$ be the subcategory of compact objects of $D_\infty(R)^\otimes$, which is formed by retracts of iterated extensions of shifts of R. Then $\mathrm{per}_\infty(R)^\otimes$ is a small E_1-monoidal unitally generated stable ∞-category.

Theorem 3.6 (Lurie [44], Proposition 7.1.2.6)**.** *The map*

$$\begin{aligned} \{E_2\text{-ring spectra}\} &\xrightarrow{\sim} \{\text{small } E_1\text{-monoidal unitally generated stable } \infty\text{-categories}\} \\ R &\longmapsto \mathrm{per}_\infty(R)^\otimes \end{aligned}$$

is an equivalence of ∞-categories.

When k is a field of characteristic zero, Kontsevich and Soibelman [38] proved that the k-linearized E_2-operad kE_2 is quasi-isomorphic to the brace operad Br. This heavily suggests that the following corollary holds.

Corollary 3.7 (* Jasso and Keller)**.** *The map*

$$\begin{aligned}\{Br_\infty\text{-algebras}\} &\xrightarrow{\sim} \{\text{small } kE_1\text{-monoidal unitally generated stable dg categories}\}\\ V &\longmapsto \mathrm{per}_{dg}(V)^{\otimes}\end{aligned}$$

is an equivalence of ∞-categories, where Br_∞ denotes homotopy Br-algebras.

As noted in Remark 1.2, the brace operad Br is a quotient of the B_∞ operad, whence each Br-algebra is also a B_∞-algebra. The content of Corollary 3.7 would yield the converse, namely that the diagram

$$\begin{array}{ccc} \{B_\infty\text{-algebras}\} & \dashrightarrow & \\ \downarrow & & \searrow \\ \{\text{small } kE_1\text{-monoidal unitally generated stable dg categories}\} & \xleftarrow{\sim} & \{Br_\infty\text{-algebras}\}\end{array}$$

induced by the maps

$$\begin{array}{ccc} V & & V \\ \downarrow & \text{and} & \searrow \\ \mathrm{per}_{dg}(V)^{\otimes} & & V/^{\mathbb{L}}(m_{i,j}, i \geq 2, j \geq 0)\end{array}$$

is commutative. This suggests the picture

$$\begin{array}{ccc} B_\infty & & \\ \downarrow & \searrow & \\ B_\infty/^{\mathbb{L}}(m_{i,j}, i \geq 2, j \geq 0) & \xrightarrow{\sim} & Br\end{array}$$

where $Br = B_\infty/(m_{i,j}, i \geq 2, j \geq 0)$ is equivalent to $B_\infty/^{\mathbb{L}}(m_{i,j}, i \geq 2, j \geq 0)$.

4. B_∞ STRUCTURES FOR SINGULARITY CATEGORIES

We now lift the B_∞ structures obtained for the Hochschild cohomology of an algebra to a categorical framework, and we use them to study several types of singularities.

4.1. **Derived categories.** The construction of the Hochschild cochain complex, together with its B_∞ structure, generalizes from k-algebras to k-categories in the sense of Mitchell [47]. A *k-category* is a category equipped with a k-module structure on each set of morphisms that is compatible with the composition (namely composing morphisms in the category is itself a k-module morphism). This can be rephrased as saying that a k-category is a category *enriched* over k-modules. We can think of k-algebras as k-categories with exactly one object, and k-categories can be seen as being k-algebras with several objects. In the sketch of the proof of Theorem 2.1 we already saw the k-category with two objects $\mathcal{G}$. Given $\mathcal{A}$ a small k-category (or a dg category in general), its *Hochschild cochain complex* $C(\mathcal{A}, \mathcal{A})$ is defined as the product total complex of the bicomplex having p-th column

$$\prod_{X_0,\dots,X_p\in\mathcal{A}} \mathrm{Hom}_k(\mathcal{A}(X_{p-1}, X_p) \otimes \cdots \otimes \mathcal{A}(X_0, X_1), \mathcal{A}(X_0, X_p))$$

*This is unpublished work in progress.

with horizontal differential

$$\begin{aligned} d(f)(a_0 \otimes \cdots \otimes a_p) =& (-1)^{|a_p||f|} a_0 f(a_n \otimes \cdots \otimes a_p) \\ &+ \sum_{i=0}^{p-1} (-1)^{s_i} f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p) \\ &+ (-1)^{s_p} f(a_0 \otimes \cdots \otimes a_{p-1}) a_p \end{aligned}$$

where $s_i = |f| - i - 1 + \sum_{j=p-i+1}^{p} |a_j|$ for $i = 0, \ldots, p-1$ are given by the Koszul sign convention. The *Hochschild cohomology* of $\mathcal{A}$ is $HH^*(\mathcal{A}) = H^*C(\mathcal{A}, \mathcal{A})$ the homology of its Hochschild cochain complex. Since the first differential in $C(\mathcal{A}, \mathcal{A})$ is given by

$$\begin{aligned} \prod_{X_0 \in \mathcal{A}} \mathcal{A}(X_0, X_0) &\longrightarrow \prod_{X_0, X_1 \in \mathcal{A}} \operatorname{Hom}_k(A(X_0, X_1), A(X_0, X_1)) \\ (\varphi_{X_0} : X_0 \to X_0) &\longmapsto (f \mapsto \varphi_{X_1} f - f \varphi_{X_0}) \end{aligned}$$

we then recover the center of the category as $HH^0(\mathcal{A}) = \operatorname{End}(\mathrm{id}_A) = Z(\mathcal{A})$ as in Section 1. Following Drinfeld [10], the notion of a derived dg category is then a sensible construction.

Theorem 4.1 (Lowen and Van den Bergh [42], Töen [53], Keller [35])**.** *Let A be a gd algebra and fix $\mathcal{U}$ a Grothendieck universe. Let* $\mathrm{Proj}(A)$ *be the category of $\mathcal{U}$-small projective right A-modules, let $D(A)$ be the unbounded derived category of A, and let $D_{dg}(A)$ be its canonical dg enhancement. Then there are canonical isomorphisms of Gerstenhaber algebras*

$$HH^*(D_{dg}(A)) \xrightarrow{\sim} HH^*(\mathrm{Proj}(A)) \xrightarrow{\sim} HH^*(A)$$

that lift to quasi-isomorphisms

$$C(D_{dg}(A), D_{dg}(A)) \xrightarrow{\sim} C(\mathrm{Proj}(A), \mathrm{Proj}(A)) \xrightarrow{\sim} C(A, A)$$

giving the equivalence of these B_∞ structures.

Remark 4.2.

(1) The isomorphism $HH^*(D_{sg}(A)) \cong HH^*(A)$ should be viewed as a derived version of the classical isomorphism $Z(\mathrm{Mod}A) \cong Z(A)$ of Section 2.
(2) In particular, we have the desirable property $Z(D_{sg}(A)) \cong Z(A)$. This does not hold without the dg enhancement, the center of the unbounded derived category $D(A)$ is in fact pathological. For example $Z(D(k[\epsilon]/(\epsilon^2))) \cong k \ltimes k^{\mathbb{N}}$ as shown by Krause and Ye [39].

4.2. **Singularity categories.** Let A be a right Noetherian k-algebra, for example a quotient of a polynomial ring $k[x_1, \ldots, x_n]/(I)$, and assume that A^e is also Noetherian. Let $\mathrm{mod}A$ be the category of finitely genrated right A-modules, let $D^b(\mathrm{mod}A)$ be its bounded derived category, and let $\mathrm{per}(A)$ be its perfect derived category (which again coincides with $\mathrm{thick}(A)$). The Verdier quotient $sg(A) = D^b(\mathrm{mod}A)/\mathrm{per}(A)$ is known as the *stable derived* category of A, employed by Buchweitz [4, 3] in the study of Cohen-Macaulay modules, or as the *singularity category* of A, rediscovered by Orlov [49] in the context of mirror symmetry. Note that when A is *smooth*, namely it has finite global dimension, then $sg(A)$ is the zero category. The *singular Hochschild cohomology* or *Tate-Hochschild cohomology* is defined as $HH^*_{sg}(A) = \operatorname{Ext}^*_{A^e}(A, A)$. As before, $HH^*_{sg}(A)$ is still graded commutative, but $sg(A^e)$ is not monoidal in an obvious way.

Theorem 4.3 (Wang [55, 54, 56]).

(1) The singular Hochschild cohomology $HH^*_{sg}(A)$ has a canonical Gerstenhaber bracket. This bracket is compatible with the graded commutative cup product, making $HH^*_{sg}(A)$ a Gerstenhaber algebra.
(2) There is a canonical cochain complex $C_{sg}(A, A)$ such that $HH^*_{sg}(A) = H^*C_{sg}(A, A)$. Moreover $C_{sg}(A, A)$ is a B_∞-algebra lifting the Gerstenhaber algebra structure on $HH^*_{sg}(A)$.

The key tool for this result is the spineless cacti operad introduced by Kaufmann [31, 33, 32, 34]. We now have a complete structural analogy between singular and classical Hochschild cohomology. This suggests that singular Hochschild cohomology may in fact be an instance of classical Hochschild cohomology.

Theorem 4.4 (Keller [36, 37]). *There is a canonical algebra morphism*

$$\Psi : HH^*_{sg}(A) \longrightarrow HH^*(sg_{dg}(A))$$

between the singular Hochschild cohomology of A and the Hochschild cohomology of the canonical dg enhancement of the singularity category of A. This morphism is usually invertible.

Seeing A as a dg category with one object, this isomorphism is given by the existence of natural dg functors

$$A \overset{i}{\rightarrowtail} D^b_{dg}(\mathrm{mod}A) \overset{p}{\twoheadrightarrow} sg_{dg}(A)$$

such that $pi \simeq 0$ in the homotopy category of dg categories. These fit in the diagram

$$\begin{array}{ccccc} D^b(\mathrm{mod}A^e) & \xrightarrow{(1\otimes i)^*} & D(A \otimes D^b_{dg}(\mathrm{mod}A)) & \xrightarrow{(i\otimes 1)^!} & D(D^b_{dg}(\mathrm{mod}A) \otimes D^b_{dg}(\mathrm{mod}A)^{op}) \\ \downarrow & & & & \downarrow {\scriptstyle (p\otimes p)^*} \\ sg(A^e) & & \dashrightarrow & & D(sg_{dg}(A) \otimes sg_{dg}(A)^{op}) \end{array}$$

where the functor

$$\begin{array}{ccc} sg(A^e) & \dashrightarrow & D(sg_{dg}(A) \otimes sg_{dg}(A)^{op}) \\ A & \longmapsto & sg_{dg}(A) \end{array}$$

induces an isomorphism of the Ext^* algebras. Unfortunately, this functor is hard to compute because it is induced by the composition of a right derived functor with a left derived functor.

Remark 4.5.

(1) The morphism $\Psi : HH^*_{sg}(A) \to HH^*(sg_{dg}(A))$ is invertible if A is commutative and the characteristic of k is zero.
(2) The morphism $\Psi : HH^*_{sg}(A) \to HH^*(sg_{dg}(A))$ is not invertible if $k \subseteq A$ is a finite inseparable field extension. In this case $HH^*_{sg}(A) \neq 0$ but $HH^*(sg_{dg}(A)) = 0$.
(3) The existence of this (iso)morphism is satisfying because $HH^*_{sg}(A)$ is computable while $HH^*(sg_{dg}(A))$ is conceptually pleasing.

Conjecture 4.6 (Keller [36, 37]). *The (iso)morphism*

$$\Psi : HH^*_{sg}(A) \longrightarrow HH^*(sg_{dg}(A))$$

lifts to an (iso)morphism of B_∞-algebras

$$\Phi : C_{sg}(A, A) \longrightarrow C(sg_{dg}(A), sg_{dg}(A)).$$

Theorem 4.7 (Chen, Li, and Wang [7]). *Let Q be a finite quiver without sinks, then Conjecture 4.6 is true for $A = kQ/(kQ_1)^2$.*

Example 4.8. *Let Q be the quiver with one vertex and one edge. Then $kQ/(kQ_1)^2 = k[\epsilon]/(\epsilon^2)$ and the isomorphism*

$$HH^*_{sg}(k[\epsilon]/(\epsilon^2)) \cong HH^*(sg_{dg}(k[\epsilon]/(\epsilon^2))).$$

can be lifted canonically to an isomorphism of B_∞-algebras.

4.3. **Reconstruction theorems for singularities.** We now apply these results to reconstruct isolated hypersurface singularities and compound Du Val singularities.

4.3.1. *Hypersurface singularities.*

Theorem 4.9 (Hua and Keller[25]). *Let $R = \mathbb{C}[[x_1, \dots, x_n]]/(f)$ be an isolated singularity. Then R is determined up to isomorphism by its Krull dimension* $\dim(R)$ *and the dg enhancement of its singularity category $sg_{dg}(R)$.*

For the sketch of the proof, set $S = \mathbb{C}[[x_1, \dots, x_n]]$ so $R = S/(f)$. We will use $S/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ the *Tyurina algebra* of R, a large enough natural number $r \in \mathbb{N}$, and a series of results to complete the diagram

$$\begin{array}{ccccc} Z(sg_{dg}(R)) & = & HH^0(sg_{dg}(R)) & \overset{\sim}{\text{---}} & HH^0_{sg}(R) \\ \downarrow \wr & & & & \wr | \\ S\Big/\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) & \overset{\sim}{\text{---}} & HH^{2r}(R) & \overset{\sim}{\text{---}} & HH^{2r}_{sg}(R) \end{array}$$

giving an isomorphism $Z(sg_{dg}(R)) \cong S/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. As noted in Remark 4.5 we have $HH^0(sg_{dg}(R)) \cong HH^0_{sg}(R)$ by Theorem 4.4. Using matrix factorization, Eisenbud [14] described a 2-periodicity that can be used to give $HH^0_{sg}(R) \cong HH^{2r}_{sg}(R)$ for $r \in \mathbb{N}$. Moreover, for large enough $r \in \mathbb{N}$ the singular Hochschild cohomology coincides with the classical Hochschild cohomology by the seminal work of Buchweitz [4, 3], giving $HH^{2r}_{sg}(R) \cong HH^{2r}(R)$. The fact that the $2r$-th degree of the Hochschild cohomology of a hypersurface is its Tyurina algebra is due to the Buenos Aires Cyclic Homology Group[22]. Given $sg_{dg}(R)$ and $\dim(R)$, our claim now follows by obtaining $Z(sg_{dg}(R))$ and applying Mather and Yau's [45] result showing that the Tyurina algebra and the Krull dimension of R suffice to determine R up to isomorphism.

4.3.2. *Compound Du Val singularities.* Let $k = \mathbb{C}$ and let R be a complete local isolated compound Du Val singularity, namely it is a three dimensional normal singularity whose generic hyperplane section is Kleinian. Set $X = \operatorname{Spec}(R)$ and let $f : Y \to X$ be a small crepant resolution, namely it is a birational resolution giving an isomorphism in codimension one, an isomorphism outside the exceptional fiber, and an equality $f^*\omega_X = \omega_Y$ where ω_X and ω_Y are the corresponding canonical divisors (in this case a resolution is small if and only if it is crepant). Let $\mathcal{F}$ be the reduced exceptional fiber of f, which is given by a tree of rational curves $\mathcal{F} = \bigcup_{i=1}^n C_i$ that is contracted to a single point by f. It has several associated dg algebras, of particular interest are *contraction algebra* Λ and its *derived contraction algebra* Γ.

Theorem 4.10 (Efimov, Lunts, and Orlov [11, 12, 13], Donovan and Wemyss [9], Laudal [40], Hua and Keller [25]).

(1) There is a canonical connective dg algebra Γ *which pro-represents the noncommutative deformations of* $\bigoplus_{i=1}^n \mathcal{O}_{C_i}$ *in* $D^b(coh(Y))$. *In particular,* $H^p\Gamma = 0$ *for all* $p \geq 0$.

(2) There is an isomorphism $H^0\Gamma \cong \Lambda$ *representing the noncommutative deformations of* $\bigoplus_{i=1}^n \mathcal{O}_{C_i}$ *in* $coh(Y)$.

Remark 4.11.

(1) The algebra Λ is finite dimensional, as is the Tyurina algebra of R, but is noncommutative. Moreover, $H^p\Gamma$ is finite dimensional for all $p \in \mathbb{Z}$.
(2) The algebra Λ determines many invariants of R, such as the width of Reid and the bidegree of the normal bundle (see Donovan and Wemyss [9]), and Katz's genus zero Gopakumar-Vafa invariants (see Toda [52] and Hua and Toda [26]).

This suggests that the algebra Λ may be enough to characterize R.

Conjecture 4.12 (Donovan and Wemyss [9]). *The derived equivalence class of* Λ *determines* R *up to isomorphism.*

Partial progress was achieved using the algebra Γ instead of Λ.

Theorem 4.13 (Hua and Keller[25]). *The algebra* Γ *is a homologically smooth bimodule 3-Calabi-Yau algebra and its derived equivalence class determines* R *up to isomorphism.*

The strategy behind the proof involves showing that there is a dg equivalence

$$sg(R) \xrightarrow{\sim} \mathcal{C}_\Gamma = \mathrm{per}(\Gamma)/D^{fd}(\Gamma)$$

between the singularity category of R and the cluster category of Γ. Here $D^{fd}(\Gamma)$ denotes the derived category of complexes M of right Γ modules whose homology H^*M has finite total dimension. The result follows by applying Theorem 4.9.

Observe that $H^*\Gamma \cong \Lambda \otimes k[u^{-1}]$ where u has degree two, whence Λ determines $H^*\Gamma$, but unfortunately Γ is not a formal dg algebra. Nevertheless, the following new approach to Conjecture 4.12 using cluster-tilting objects keeps our hopes alive. Consider the projection functor $p : \mathrm{per}(V) \to \mathcal{C}_\Gamma$ and let $T = p(\Gamma)$. It was proven by Amiot [1] that T is a $2\mathbb{Z}$ cluster-tilting object in the sense of Iyama and Yoshino [28] and Geiss, Keller, and Oppermann [15], namely

$$\mathrm{add}(T) = \{X \in \mathcal{C}_\Gamma | \mathrm{Ext}^i(T, X) = 0 \forall i \notin 2\mathbb{Z}\} = \{X \in \mathcal{C}_\Gamma | \mathrm{Ext}^i(X, T) = 0 \forall i \notin 2\mathbb{Z}\}$$

where $\mathrm{add}(T)$ is the smallest subcategory of $\mathcal{C}_\Gamma$ that is closed under finite direct sums, closed under retracts, and contains T. Moreover, we have $\Lambda = H^0\Gamma \cong \mathrm{End}(T)$ where the endomorphisms are not at the derived level. Under this perspective Conjecture 4.12 is implied by the following more general statement.

Conjecture 4.14. *Let* $\mathcal{C}$ *be a dg enhanced triangulated category satisfying suitable technical conditions. If* $\mathcal{C}$ *contains a* $2\mathbb{Z}$*-cluster-tilting object* T *then the non-derived endomorphisms* $\mathrm{End}_{\mathcal{C}}(T)$ *determine* $\mathcal{C}$ *up to quasi-equivalence of dg categories.*

This would yield the surprising fact that the higher structure of the category $\mathcal{C}$, namely its dg enhancement, is completely determined by its lower structure, namely the non-derived endomorphisms $\mathrm{End}_{\mathcal{C}}(T)$. Assuming Conjecture 4.14 then Conjecture 4.12 follows by applying Theorem 4.9.

$$\Lambda = \mathrm{End}_{\mathcal{C}_\Gamma} \xrightarrow{\text{Conjecture 4.14}} (\mathcal{C}_\Gamma)_{dg} \cong sg_{dg}(R) \xrightarrow{\text{Theorem 4.9}} R \text{ up to isomorphism}$$

Remark 4.15. There are a couple of reasons to hope that Conjecture 4.14 holds.

(1) Recent work of Muro [48] shows that when $\mathcal{C}$ is a dg enhanced triangulated category (satisfying suitable technical conditions) containing a $1\mathbb{Z}$-cluster-tilting-object T, then $\mathcal{C}$ is determined by the non-derived endomorphisms $\mathrm{End}_{\mathcal{C}}(T)$ up to quasi-equivalence of dg categories. A family of categories containing $1\mathbb{Z}$-cluster-tilting-objects are $sg(R)$ for R a simple singularity of even dimension.

(2) A surprising feature of Iyama's [27] higher homological theory is that many phenomena occurring in dimension one generalize to higher dimensions. It is reasonable to expect this to continue being the case.

Even more recently Jasso and Muro [29] showed that when $\mathcal{C}$ is a dg enhanced triangulated category (under suitable technical conditions) containing a $d\mathbb{Z}$-cluster-tilting-object T, then $\mathcal{C}$ is determined by the derived endomorphisms $\mathcal{R}\mathrm{End}_{\mathcal{C}}(T)$ up to quasi-equivalence of dg categories. In an appendix to their work, Keller proved Conjecture 4.12.

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