Triality: a particularity of $Spin(8)$

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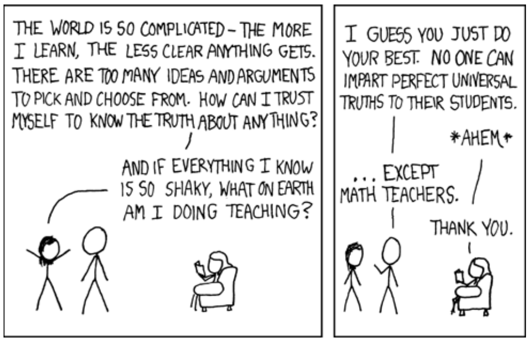
THE WORLD IS SO COMPLICATED – THE MORE I LEARN, THE LESS CLEAR ANYTHING GETS. THERE ARE TOO MANY IDEAS AND ARGUMENTS TO PICK AND CHOOSE FROM. HOW CAN I TRUST MYSELF TO KNOW THE TRUTH ABOUT ANYTHING?
AND IF EVERYTHING I KNOW IS SO SHAKY, WHAT ON EARTH AM I DOING TEACHING?
I GUESS YOU JUST DO YOUR BEST. NO ONE CAN IMPART PERFECT UNIVERSAL TRUTHS TO THEIR STUDENTS.
AHEM
... EXCEPT MATH TEACHERS.
THANK YOU.

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1 Introduction

About a year ago, when having to choose a mathematical subject in which to deepen, I was immediately attracted by the subject known as *Lie Theory*. The way topology, algebra and analysis interrelate to make a theoretically interesting theory yet extremely useful struck me as something that ought to be explored. I was fortunate to have Professor Jaume Aguadé to guide me in this scouting, since my first attempts were quite poor. He introduced me to the wonders of $\mathrm{Spin}(n)$ through the eyes of Frank Adams, and what you now have in your hands is an attempt to understand this point of view.

The interest of Adams' way of tackling the construction of a triality lies beyond the academic. Instead of rejoicing in complications and looking for inspiration among the exceptional Jordan algebras, Cayley's projective plane or the exceptional Lie group F_4, he takes the Dynkin diagrams as a basis. Undoubtedly, the relation among the outer automorphisms and the permutations of the representations of $\mathrm{Spin}(8)$ provides a simpler and more elementary explanation of the phenomenon.

An outline of the Chapters follows:

Chapter 2 is meant to be a gentle but broad overview of the mathematical objects that will be used throughout the work. These include the **general linear groups** $\mathrm{GL}_n(\mathbb{K})$, the **orthogonal groups** $\mathrm{O}(n)$ and the **representations of a group** $(\mathbb{V}, \mu)$.

Chapter 3 is where the **Clifford algebras** Cl_n and the **Clifford groups** Γ_n are defined. The main structural properties of the first object are analysed, whereas over the second object we expand on how a conjugation defines an **isometry** in a way that establishes an intimate relation with $\mathrm{O}(n)$.

Chapter 4 contains the remaining of the construction of the **pin groups** $\mathrm{Pin}(n)$ and **spinor groups** $\mathrm{Spin}(n)$. This is followed by the pertinent explanation of the claim that the second is a **double cover** of $\mathrm{SO}(n)$ as well as the exposition of the main properties their representations enjoy.

Chapter 5 is where the main interest of the work yields, as it is devoted to **triality** and how it relates to $\mathrm{Spin}(8)$. This includes characterizing the **outer automorphisms** of $\mathrm{Spin}(8)$, defining the concept of triality and, using its relation with **duality**, understand how a triality can be induced by $\mathrm{Spin}(8)$.

Due to the subjective nature of the approach taken in the work, many results that are to be needed must be provided without a proof. An effort has been made to provide references for all of them. The work is also wished to provide a more topologically flavoured approach, and thus a few results that are not essential to the core of the work but broaden its scope are also explained. However, since they were the author's at the start of the journey, only basic topology and algebra are taken as prerequisites, and every relevant definition has been included.

2 Basic Ideas and Concepts

As the name of the section suggests, here we present a rough sketch of the basic ideas and concepts that will be used throughout the work and that motivate the study.

2.1 The Classical Groups

In this section we present a brief introduction to the **Classical groups**, which are the groups that arise from considering matrices with entries in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, although what we expose is mainly valid in general for any field $\mathbb{K}$.

In the whole work, given $m, n \in \mathbb{N}$, the set of $m \times n$ matrices with entries on $\mathbb{K}$ is denoted by $\mathrm{M}_{m\times n}(\mathbb{K})$, and the special case when $m = n$ is denoted as $\mathrm{M}_n(\mathbb{K})$. It is well known that $\mathrm{M}_n(\mathbb{K})$ is a $\mathbb{K}$ vector space with the operations of matrix addition and scalar multiplication, the zero vector being O_n the matrix with all entries zero.

Definition 1. *We define the* general linear group *as the group:*

$$\mathrm{GL}_n(\mathbb{K}) = \{A \in \mathrm{M}_n(\mathbb{K}) : \det A \neq 0\},$$

and the special linear group *as the group:*

$$\mathrm{SL}_n(\mathbb{K}) = \{A \in \mathrm{M}_n(\mathbb{K}) : \det A = 1\}.$$

These sets are easily checked for begin groups as claimed. However, this is not enough, since we also need to consider the topology of the classical groups. This would mean that $\mathrm{SL}_n(\mathbb{K}) \leq \mathrm{GL}_n(\mathbb{K})$ is in fact what will be called a **matrix group**. Said topology is usually taken only over $\mathbb{R}$ or $\mathbb{C}$ and is achieved by considering them metric spaces. Since $\mathrm{M}_n(\mathbb{K})$ is a metric space of dimension n^2 over $\mathbb{K}$, we may define a norm based on the existent norm over $\mathbb{K}^n$.

Definition 2. *For any vector* $\mathbf{x} \in \mathbb{K}^n$*, say* $\mathbf{x} = (x_1, \dots, x_n)$*, we define the* norm of the vector *as:*

$$|\mathbf{x}| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

For any matrix $A \in \mathrm{M}_n(\mathbb{K})$*, we define the* norm of the matrix *as:*

$$||A|| = \sup\left\{\frac{|Ax|}{|x|} : 0 \neq \mathbf{x} \in \mathbb{K}^n\right\}.$$

It is easy to check that $|\ \ |$ is a norm in $\mathbb{K}^n$ and $||\ \ ||$ is a norm in $\mathrm{M}_n(\mathbb{K})$, and as in a finite dimensional vector space all norms are equivalent, we shall use the one that suits our needs.

Definition 3. *A subgroup* $G \leq \mathrm{GL}_n(\mathbb{K})$ *which is also a closed subspace is a* matrix group *(over* $\mathbb{K}$*). We may say that* G *is a* matrix subgroup *of* $\mathrm{GL}_n(\mathbb{K})$*.*

We now proceed to define the most important subgroups of $\mathrm{GL}_n(\mathbb{K})$, which altogether form the classical subgroups. The following definitions may be taken as a dictionary or preamble. For a more detailed coverage of the metric and/or topological aspects of the classical groups, see [3, Chapter 1], [5, Part 2], [4, Part A] or [9].

Definition 4. *We define the* affine group *in dimension $n \in \mathbb{N}$ (over $\mathbb{K}$) as:*

$$\mathrm{Aff}_n(\mathbb{K}) = \left\{ \begin{bmatrix} A & \mathbf{t} \\ 0 & 1 \end{bmatrix} : A \in \mathrm{GL}_n(\mathbb{K}), \mathbf{t} \in \mathbb{K}^n \right\}.$$

We define the translation subgroup *of $\mathrm{Aff}_n(\mathbb{K})$ as:*

$$\mathrm{Trans}_n(\mathbb{K}) = \left\{ \begin{bmatrix} \mathrm{Id}_n & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} : \mathbf{t} \in \mathbb{K}^n \right\}.$$

In this context, very important are matrices that have their transpose as their inverse: $A \in \mathrm{M}_n(\mathbb{K})$ with $A^T A = \mathrm{Id}_n = AA^T$, the *orthogonal* matrices.

Definition 5. *We define the (real)* orthogonal group *in dimension $n \in \mathbb{N}$ as:*

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = \mathrm{Id}_n\}.$$

We define the positive *and* negative *components of the orthogonal group as:*

$$\begin{aligned} \mathrm{O}(n)^+ &= \{A \in \mathrm{O}(n) : \det A = 1\}, \\ \mathrm{O}(n)^- &= \{A \in \mathrm{O}(n) : \det A = -1\}, \end{aligned}$$

respectively. The positive component is also named the special orthogonal group $\mathrm{SO}(n)$*.*

We obviously have $\mathrm{O}(n) = \mathrm{O}(n)^+ \cup \mathrm{O}(n)^-$ and $\mathrm{O}(n)^+ \cap \mathrm{O}(n)^- = \emptyset$. The orthogonal and their special orthogonal subgroup are studied because of the relationship they have with *isometries*, that is, distance preserving bijections ($f : \mathbb{R}^n \to \mathbb{R}^n$ bijection such that $|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$):

Proposition 1. *If $A \in \mathrm{GL}_n(\mathbb{R})$, then the following conditions are equivalent:*

1. *A is a linear isometry.*
2. *$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.*
3. *$A^T A = \mathrm{Id}_n$, that is, A is orthogonal.*

Proof. It can be found in [3, Proposition 1.38]. □

In some bibliography, this result gives ground to elements of $\mathrm{SO}(n)$ being called *direct isometries* or *rotations* and elements of $\mathrm{O}(n)^-$ being called *indirect isometries*. This in particular means that there is a full isometry group.

Definition 6. *We define the full* isometry group *of dimension $n \in \mathbb{N}$ as:*

$$\mathrm{Isom}_n(\mathbb{R}) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^n : f \text{ is an isometry}\}.$$

Which contains $\mathrm{Trans}_n(\mathbb{R})$ and in fact $\mathrm{Isom}_n(\mathbb{R}) \leq \mathrm{Aff}_n(\mathbb{R})$ is a matrix subgroup.

Elements of $\mathrm{O}(n)$ determine certain important structures. A subspace $H \subseteq \mathbb{R}^n$ of dimension $n-1$ is usually called a *hyperplane* in $\mathbb{R}^n$, which has associated a linear transformation $\theta_H : \mathbb{R}^n \to \mathbb{R}^n$ called the *reflection in the hyperplane.* This mapping is characterized by the fact that every element $\mathbf{x} \in \mathbb{R}^n$ can be uniquely expressed as $\mathbf{x} = \mathbf{x}_H + \mathbf{x'}_H$ with $\mathbf{x}_H \in H$ and $\mathbf{y} \cdot \mathbf{x'}_H = 0$ for every $\mathbf{y} \in H$, and thus we may set:

$$\begin{array}{rccc} \theta_H & : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & \mathbf{x} & \longmapsto & \mathbf{x}_H - \mathbf{x'}_H \end{array}$$

Lemma 1. *For a hyperplane $H \subseteq \mathbb{R}^n$, the reflection in the hyperplane θ_H is an indirect isometry of $\mathbb{R}^n$, in particular $\theta_H \in \mathrm{O}(n)$.*

Proof. It can be found in [3, Lemma 1.40]. □

A particular case of reflection in the hyperplane is when it is with respect to the standard basis of $\mathbb{R}^n$, because in this way it may be written as:

$$P \begin{bmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & -1 \end{bmatrix} P^T,$$

for some $P \in \mathrm{O}(n)$. In fact, these generate $\mathrm{O}(n)$.

Proposition 2. *Every element $A \in \mathrm{O}(n)$ is a product of hyperplane reflections. The number of these is even if $A \in \mathrm{SO}(n)$ and odd if $A \in \mathrm{O}(n)^-$.*

Proof. It can be found in [3, Lemma 1.41]. □

Definition 7. *We define the* standard block *as:*

$$J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Let $m \geq 1$, we define:

$$J_{2m} = \begin{bmatrix} J & O_2 & \cdots & O_2 \\ O_2 & J & \cdots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \cdots & J \end{bmatrix},$$

a non-degenerate skew symmetric matrix. We define the (real) symplectic group *as:*

$$\mathrm{Symp}_{2m} = \{A \in \mathrm{GL}_{2m}(\mathbb{R}) : A^T J_{2m} A = J_{2m}\}.$$

This was the last of the real classical groups we consider here. We now switch our attention to matrices with entries in the complex numbers. Over such matrices not only we have a natural extension of the dot product (which is not linear but sesquilinear) with $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} \overline{x_i} y_i$ for $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, but we also have the additional option of conjugating their entries, and thus computing the *hermitian conjugate* of any matrix $A \in \mathrm{GL}_n(\mathbb{C})$ as $A^* = (\overline{A})^T = \overline{A^T}$.

Definition 8. *We define the* unitary group *in dimension $n \in \mathbb{N}$ as:*

$$\mathrm{U}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^* A = \mathrm{Id}_n\}.$$

We define the special unitary group *as:*

$$\mathrm{SU}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^* A = \mathrm{Id}_n, \det A = 1\}.$$

2.2 Continuous Group Actions

In this section we shall consider **group actions**, which are a fundamental concept in ordinary group theory. These actions will help us define the concept of **representation**, the basis of all our work relating triality.

Definition 9. *Let G be a topological space and consider $G \times G$ as the product space. Suppose that G is also a group with a multiplication map $G \times G \to G$ and inverse map $G \to G$. We say that G is a* topological group *if the multiplication and the inverse are continuous.*

There are many examples of topological groups, the simplest ones are obtained from arbitrary groups given discrete topologies. The first relevant examples are $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ with the matrix multiplication and inverse maps and the subspace topologies inherited from $\mathrm{M}_n(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{C})$ accordingly. These examples are still relatively simple when compared to what can be found.

Definition 10. *Given $\mathbb{K}$ a field and $(\mathbb{V}, |\ \ |)$ a finite dimensional normed $\mathbb{K}$ vector space, we define:*

$$\mathrm{End}_{\mathbb{K}}(\mathbb{V}) = \{f : \mathbb{V} \longrightarrow \mathbb{V} : f \textit{ is a linear transformation}\}.$$

We define the general linear group of V *as:*

$$\mathrm{GL}_{\mathbb{K}}(\mathbb{V}) = \{f : \mathbb{V} \longrightarrow \mathbb{V} : f \textit{ is an ivertible linear transformation}\}.$$

This space is equipped with the standard *operator norm* inherited from the vector space $\mathbb{V}$, that is $||f|| = \sup\{|f(v)| : v \in \mathbb{V}, |v| = 1\}$. Moreover, this definition clearly generalises the general linear matrix group.

Definition 11. *An* action μ *of a group G on a set X is a function:*

$$\begin{array}{cccc} \mu & : & G \times X & \longrightarrow & X \\ & & (g, x) & \longmapsto & gx \end{array}$$

satisfying:

1. *$(gh)x = g(hx)$, that is, $\mu(gh, x) = \mu(g, \mu(h, x))$ for all $g, h \in G$, $x \in X$,*

2. *$ex = x$ for all $x \in X$ with $e \in G$ being the identity element.*

An action has two important sets associated to it:

Definition 12. *Consider G a group acting on a set X. Let $x \in X$, we define the* stabiliser *of x as:*

$$\mathrm{Stab}_G(x) = \{g \in G : gx = x\}.$$

We define the orbit *of x as:*

$$\mathrm{Orb}_G(x) = \{gx \in X : g \in G\}.$$

And if the group we are considering is a topological one, this notion of action can be generalised.

Definition 13. *Let G be a topological group and X be a topological space, consider $G \times X$ with the product topology. Then a group action $\mu : G \times X \to X$ is a* continuous group action *if the function μ is continuous.*

We are particularly interested in actions over $(\mathbb{V}, |\ \ |)$, when G is a certain matrix group.

Definition 14. *Let $(\mathbb{V}, |\ \ |)$ be a finite dimensional normed $\mathbb{K}$ vector space, G be a matrix group that has a continuous homomorphism $\varphi : G \to \mathrm{GL}_{\mathbb{K}}(\mathbb{V})$. The associated action:*

$$\begin{array}{rccc} \mu_\varphi & : & G \times \mathbb{V} & \longrightarrow & \mathbb{V} \\ & & (g, v) & \longmapsto & \varphi(g)(v) \end{array}$$

is called a (continuous) linear action *or* representation *of G on $\mathbb{V}$.*

This is well defined because the associated action μ_φ is based on $\varphi(g) \in \mathrm{GL}_{\mathbb{K}}(\mathbb{V})$ which is linear and thus in finite dimensional cases immediately continuous. In this case, by choosing a basis for $\mathbb{V}$, we may as well assume that $\mathbb{V} = \mathbb{K}^n$ and thus a continuous group action is essentially the same thing as a continuous group homomorphism $\varphi : G \to \mathrm{GL}_n(\mathbb{K})$. Note that we may make the slight abuse of notation of interpreting a representation as the vector space the group acts on.

Definition 15. *Let $\mathbb{V}$ be a non trivial representation of a group G. We say that $\mathbb{V}$ is* irreducible *if it has no proper G submodules other than the trivial.*

It is a natural thing to wonder when two representations should be considered the same. A natural way is to consider this when there exist analogues of **equivariant maps**.

Definition 16. *Let G be a group and X, Y two sets where G acts on. A map $f : X \to Y$ is said to be* equivariant *when for every $g \in G$ it holds $f(g \cdot x) = g \cdot f(x)$.*

Definition 17. *Let* $(\mathbb{V}_1, \mu_1)$, $(\mathbb{V}_2, \mu_2)$ *be two representations of a group* G. *A* $\mathbb{K}$ *linear map* $F : \mathbb{V}_1 \to \mathbb{V}_2$ *is called an* intertwining operator *between* μ_1 *and* μ_2 *when for every* $g \in G$ *it holds* $F \circ \mu_1(g) = \mu_2(g) \circ F$, *that is, the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{V}_1 \arrow[r, "F"] \arrow[d, "\mu_1"'] & \mathbb{V}_2 \arrow[d, "\mu_2"] \\
\mathbb{V}_1 \arrow[r, "F"'] & \mathbb{V}_2
\end{tikzcd} .$$

We say that μ_1 *and* μ_2 *are* equivalent *as representations if* F *is an isomorphism.*

Observation 1. *Notice how given* $(\mathbb{V}, \mu_1)$ *and* $(\mathbb{V}, \mu_2)$ *two representations of a group* G *(that is* $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}$*), we may say that* μ_1 *and* μ_2 *are equivalent if and only if there exists* $A \in \mathrm{GL}_{\mathbb{K}}(\mathbb{V})$ *so that for every* $g \in G$*:*

$$\mu_1(g) = A^{-1}\mu_2(g)A.$$

3 The Clifford Algebras

In this section we present the **real Clifford algebras**, the generalisation of the sequence of Real algebras that start with $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. We will need them to define the main object of interest of this work: the Spin(8) group.

Definition 18. *We define the* real Clifford algebra *in $n \in \mathbb{N}$ variables Cl_n as the $\mathbb{R}$ algebra generated by the elements $e_1, \ldots, e_n \in \mathrm{Cl}_n$ for which:*

$$\begin{cases} e_s e_r = -e_r e_s \text{ if } s \neq r, \\ e_r^2 = -1. \end{cases}$$

These algebras are finite dimensional:

Proposition 3. *The real Clifford algebra Cl_n has an $\mathbb{R}$ basis:*

$$\mathcal{B}_n = \{e_{i_1} \cdots e_{i_r} : 1 \leq i_1 < \cdots < i_r \leq n \text{ and } 0 \leq r \leq n\},$$

with $e_{i_1} \cdots e_{i_r} = 1$ for $r = 0$ by convention.

Proof. Clearly, every $x \in \mathrm{Cl}_n$ can be written as an expansion with certain coefficients of the elements in $\mathcal{B}_n$. We will now prove by induction that these elements are linearly independent. This is clear in the first two cases, since $\mathrm{Cl}_0 = \langle\{1\}\rangle_{\mathbb{R}}$ and $\mathrm{Cl}_1 = \langle\{1, e_1\}\rangle_{\mathbb{R}}$ because they are the vector spaces of dimension 1 and 2. Assume $\mathcal{B}_{n-1}$ is a basis of Cl_{n-1}, consider:

$$\mathcal{B}_n = \{e_{i_1} \cdots e_{i_r} e_n : 1 \leq i_1 < \cdots < i_r \leq n-1 \text{ and } 0 \leq r \leq n-1\} \cup \mathcal{B}_{n-1}.$$

Clearly the elements in $\mathcal{B}_{n-1}$ are linearly independent among them. As we have e_n multiplying in the first set, we have that the first and second sets are linearly independent. It is thus enough to see that the rest of them are also linearly independent among them. This is true because any linear sum has e_n as a common factor:

$$\left(\sum_{i_1,\ldots,i_r} \lambda_{i_1,\ldots,i_r} e_{i_1} \cdots e_{i_r} \right) e_n = \sum_{i_1,\ldots,i_r} \lambda_{i_1,\ldots,i_r} e_{i_1} \cdots e_{i_r} e_n = 0,$$

and as we are in a vector space, the remaining sum must be zero. Since this sum is of elements of $\mathcal{B}_{n-1}$, we must have that all the coefficients are zero, obtaining the desired result. □

This means that $\dim_{\mathbb{R}} \mathrm{Cl}_n = 2^n$, as we are essentially taking the power set of $\{1, \ldots, n\}$. Having such basis allow us to readily calculate:

Example 1. *Let $i, j, k, l \in \{1, \ldots, n\}$ be distinct numbers. Then:*

$$e_i e_j e_k e_l = -e_i e_k e_j e_l = -e_k e_i e_l e_j = e_k e_l e_i e_j.$$

This definition generalises the real algebras, as we wished.

Proposition 4. *There are isomorphisms of* $\mathbb{R}$ *algebras* $\mathrm{Cl}_0 \cong \mathbb{R}$, $\mathrm{Cl}_1 \cong \mathbb{C}$ *and* $\mathrm{Cl}_2 \cong \mathbb{H}$.

Proof. For Cl_0 the result is obvious, as in fact $\mathrm{Cl}_0 = \mathbb{R}$.

For Cl_1, the function:

$$\begin{array}{ccc} \mathrm{Cl}_1 & \longrightarrow & \mathbb{C} \\ x + ye_1 & \longmapsto & x + yi \end{array}$$

with $x, y \in \mathbb{R}$, is an $\mathbb{R}$ linear ring isomorphism.

For Cl_2, the function:

$$\begin{array}{ccc} \mathrm{Cl}_2 & \longrightarrow & \mathbb{H} \\ t + xe_1 + yr_2 + ze_1e_2 & \longmapsto & t + xi + yj + zk \end{array}$$

with $t, x, y, z \in \mathbb{R}$, is an $\mathbb{R}$ linear ring isomorphism. □

Moreover, we can define an ordering for the monomials in the base of Cl_n: we associate to the basis element $e_{i_1} \cdots e_{i_r}$ the number:

$$1 + 2^{i_1 - 1} + \cdots + 2^{i_r - 1},$$

taking it as 1 when $r = 0$. As every integer m such that $1 \leq m \leq 2^n$ has a unique binary expansion:

$$m = m_0 + 2m_1 + \cdots + 2^n m_n \text{ with } m_i = 0, 1 \text{ for } i = 1, \ldots, n,$$

this provides a one to one correspondence between the numbers m and the basis monomials of Cl_n. Using the left regular representation over $\mathbb{R}$ associated to the basis of Cl_n, we can realise Cl_n as a subalgebra of $\mathrm{M}_{2^n}(\mathbb{R})$.

The **universal property** that characterizes the Clifford algebras Cl_n is based on the observation that there is an $\mathbb{R}$ linear transformation:

$$\begin{array}{cccc} j_n & : & \mathbb{R}^n & \longrightarrow & \mathrm{Cl}_n \\ & & \sum_{r=1}^n x_r \mathbf{e}_r & \longmapsto & \sum_{r=1}^n x_r e_r \end{array}$$

for which:

$$j_n(\mathbf{x})^2 = j_n \left(\sum_{r=1}^n x_r \mathbf{e}_r \right)^2 = - \sum_{r=1}^n x_r^2 = - \left| \sum_{r=1}^n x_r \mathbf{e}_r \right|^2 = -|\mathbf{x}|^2,$$

for $\mathbf{x} \in \mathbb{R}^n$.

Theorem 1 (Universal property of a Clifford algebra)**.** *Let* A *be an* $\mathbb{R}$ *algebra. If* $f : \mathbb{R}^n \to A$ *is an* R *linear transformation for which* $f(\mathbf{x})^2 = -|x|^2 1$, *then there is a unique homomorphism of* $\mathbb{R}$ *algebras* $F : \mathrm{Cl}_n \to A$ *for which* $F \circ j_n = f$*:*

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[d, "j_n"'] & A . \\ \mathrm{Cl}_n \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

Proof. We define the homomorphism:

$$\begin{array}{rccc} F & : & \mathrm{Cl}_n & \longrightarrow & A \\ & & e_r & \longmapsto & f(\mathbf{e}_r) \end{array}$$

which extends to a ring homomorphism given on the basis monomials by:

$$\begin{array}{rccc} F & : & \mathrm{Cl}_n & \longrightarrow & A \\ & & e_{i_1}\cdots e_{i_r} & \longmapsto & f(\mathbf{e}_{i_1})\cdots f(\mathbf{e}_{i_r}) \end{array}$$

which as is usual with these constructions, fulfils the conditions and is unique by exigences of the construction. □

Corollary 1. *Let U be an $\mathbb{R}$ algebra and $j : \mathbb{R}^n \to U$ be an R linear transformation for which $j((x))^2 = -|\mathbf{x}|^2 1$. Suppose that U and j have the universal property enjoyed by Cl_n and j_n. Then there is a unique $\mathbb{R}$ algebra isomorphism $\Phi : \mathrm{Cl}_n \to U$ which satisfies $\Phi \circ j_n = j$.*

Proof. It is immediate by using the universal property of Cl_n. □

Example 2. *The first 8 real Clifford algebras are:*

Clifford algebra	Matrix group	Dimension
Cl_0	$\mathbb{R}$	1
Cl_1	$\mathbb{C}$	2
Cl_2	$\mathbb{H}$	4
Cl_3	$\mathbb{H} \times \mathbb{H}$	8
Cl_4	$\mathrm{M}_2(\mathbb{H})$	16
Cl_5	$\mathrm{M}_4(\mathbb{C})$	32
Cl_6	$\mathrm{M}_8(\mathbb{R})$	64
Cl_7	$\mathrm{M}_8(\mathbb{R}) \times \mathrm{M}_8(\mathbb{R})$	128
Cl_8	$\mathrm{M}_{16}(\mathbb{R})$	256

Observation 2. *We may consider the $\mathbb{R}$ linear transformation:*

$$\begin{array}{rccc} \alpha_0 & : & \mathbb{R}^n & \longrightarrow & \mathrm{Cl}_n \\ & & \mathbf{x} & \longmapsto & -j_n(\mathbf{x}) \end{array}$$

where we have $\alpha_0(\mathbf{x}) = j_n(-\mathbf{x})$ and $\alpha_0(\mathbf{x})^2 = j_n(-\mathbf{x})^2 = -|\mathbf{x}|^2$. By Theorem 1, there is a unique algebra homomorphism $\alpha : \mathrm{Cl}_n \to \mathrm{Cl}_n$ for which $\alpha(j_n(\mathbf{x})) = \alpha_0(\mathbf{x})$. In particular:

$$\alpha(e_r) = \alpha(j_n(\mathbf{e_r})) = \alpha_0(\mathbf{e_r}) = -j_n(\mathbf{e_r}) = -e_r.$$

Since α is an homomorphism, given $1 \leq r \leq n$ and $1 \leq i_1 < \cdots < i_r \leq n$, for the associated term of the basis we have:

$$\alpha(e_{i_1}\cdots e_{i_r}) = \alpha(e_{i_1})\cdots\alpha(e_{i_r}) = (-1)^r e_{i_1}\cdots e_{i_r} = \begin{cases} e_{i_1}\cdots e_{i_r} & \text{if } r \text{ is even,} \\ -e_{i_1}\cdots e_{i_r} & \text{if } r \text{ is odd.} \end{cases} \tag{1}$$

As we have the identity when r is even and an inversion when r is odd, clearly α is its own inverse and thus it is an isomorphism.

Definition 19. *The* canonical automorphism *of* Cl_n *is the homomorphism* $\alpha : \mathrm{Cl}_n \to \mathrm{Cl}_n$ *defined over the basis of* Cl_n *by Equation* (1)*.*

As a noteworthy point that will not be discussed further, because of the following periodicity result, these determine all the Clifford algebras.

Theorem 2. *For* $n \in \mathbb{N}$*, we have* $\mathrm{Cl}_{n+8} \cong \mathrm{Cl}_n \otimes \mathrm{M}_{16}(\mathbb{R})$*.*

Proof. A comprehensive exposition can be found in [10]. For a KO-theory of real vector bundles approach see [11]. □

As the reader may have already noticed, this same construction can also be considered with the complex numbers $\mathbb{C}$, obtaining the **complex Clifford algebras** $\mathrm{Cl}_n(\mathbb{C}) = \mathrm{Cl}_n \otimes \mathbb{C}$. In this case, the periodicity result is even more deterministic than before, as the first algebra is enough to generate the rest.

Theorem 3. *For* $n \in \mathbb{N}$*, we have that* $\mathrm{Cl}_{n+2} \otimes \mathbb{C} \cong (\mathrm{Cl}_n \otimes \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{M}_2(\mathbb{C})$*.*

Proof. A general proof can be found in [12]. □

Example 3. *The first* 2 *complex Clifford algebras are:*

Clifford algebra	Matrix group	Dimension
Cl_0	$\mathbb{C}$	2
Cl_1	$\mathbb{C} \oplus \mathbb{C}$	4

This introduces the question of the structure on Cl_n. We start by defining a **conjugation** over Cl_n.

Definition 20. *We define the* conjugation *over the basis of* Cl_n *as the mapping:*

$$\begin{array}{cccc} \overline{(\ )} : & \mathrm{Cl}_n & \longrightarrow & \mathrm{Cl}_n \\ & \overline{e_{i_1} \cdots e_{i_r}} & \longmapsto & (-1)^r e_{i_r} \cdots e_{i_1} \end{array}$$

where $1 \leq r \leq n$*,* $1 \leq i_1 < \cdots < i_r \leq n$*, and satisfying:*

$$\overline{u+v} = \overline{u} + \overline{v}, \quad \overline{tu} = t\overline{u},$$

for $u, v \in \mathrm{Cl}_n$ *and* $t \in \mathbb{R}$*.*

Observation 3. *The conjugation is not a ring homomorphism for* $n > 1$*, since whenever* $r < s$ *we have:*

$$\overline{e_r e_s} = e_s e_r = -e_r e_s = -\overline{e_r}\,\overline{e_s} \neq \overline{e_r}\,\overline{e_s}.$$

However, it is a ring anti-homomorphism as for all $u, v \in \mathrm{Cl}_n$:

$$\overline{uv} = \overline{v}\,\overline{u}.$$

This result is clear over the basis of Cl_n, which generalises directly to linear combinations by definition of conjugation.

When $n = 1, 2$ the conjugation defined agrees with the conjugations in $\mathbb{C}$ and $\mathbb{H}$.

The fact that α behaves either as the identity or as an inversion means we can differentiate two classes on Cl_n.

Definition 21. *We define a* $\pm$-grading *on* Cl_n *via:*

$$\mathrm{Cl}_n^+ = \{u \in \mathrm{Cl}_n : \alpha(u) = u\}, \quad \mathrm{Cl}_n^- = \{u \in \mathrm{Cl}_n : \alpha(u) = -u\}.$$

Proposition 5. *1. Every element $u \in \mathrm{Cl}_n$ can be uniquely expressed as $u = u^+ + u^-$ with $u^+ \in \mathrm{Cl}_n^+$ and $u^- \in \mathrm{Cl}_n^-$. Thus $\mathrm{Cl}_n = \mathrm{Cl}_n^+ \oplus \mathrm{Cl}_n^-$.*

2. This decomposition is multiplicative:

$$\begin{cases} uv \in \mathrm{Cl}_n^+ \text{ if } u, v \in \mathrm{Cl}_n^+ \text{ or } u, v \in \mathrm{Cl}_n^-, \\ uv, vu \in \mathrm{Cl}_n^- \text{ if } u \in \mathrm{Cl}_n^+ \text{ and } v \in \mathrm{Cl}_n^-. \end{cases}$$

Proof. 1. Consider:

$$u^+ = (u + \alpha(u))/2, \quad u^- = (u - \alpha(u))/2.$$

They satisfy $\alpha(u^+) = u^+$ and $\alpha(u^-) = -u^-$, so that $u^+ \in \mathrm{Cl}_n^+$ and $u^- \in \mathrm{Cl}_n^-$, with $u = u^+ + u^-$. Take $u = u_+ + u_-$ with $u_+ \in \mathrm{Cl}_n^+$ and $u_- \in \mathrm{Cl}_n^-$ another decomposition, then as $\alpha(u) = u_+ - u_-$:

$$(u + \alpha(u))/2 = u_+, \quad (u - \alpha(u))/2 = u_-,$$

and the decomposition is unique, defining the vector space direct sum decomposition.

2. Let $uv \in \mathrm{Cl}_n^+$. As α is a ring homomorphism:

$$uv = \alpha(uv) = \alpha(u)\alpha(v),$$

which means that u and v have the same $\pm$-grading and thus $u, v \in \mathrm{Cl}_n^+$ or $u, v \in \mathrm{Cl}_n^-$.

Let $uv, vu \in \mathrm{Cl}_n^-$. As α is a ring homomorphism:

$$uv = -\alpha(uv) = -\alpha(u)\alpha(v), \quad vu = -\alpha(vu) = -\alpha(v)\alpha(u),$$

which means that in both cases u and v have different $\pm$-grading and thus $u \in \mathrm{Cl}_n^+$ and $v \in \mathrm{Cl}_n^-$.

□

Observation 4. *For the vector spaces* $\mathrm{Cl}_n^{\pm}$ *we have two basis that consist of the monomials:*

$$\begin{cases} e_{i_1}\cdots e_{i_{2m}} \in \mathrm{Cl}_n^+ \text{ for } 1 \le i_1 < \cdots < i_{2m} \le n, \\ e_{i_1}\cdots e_{i_{2m+1}} \in \mathrm{Cl}_n^- \text{ for } 1 \le i_1 < \cdots < i_{2m+1} \le n. \end{cases}$$

This is proven in the same fashion Proposition 3 was.

This results in a canonical isomorphism between Cl_n *and* Cl_{n+1}^+*:*

$$\begin{array}{rccc} \Phi & : & \mathrm{Cl}_n & \longrightarrow & \mathrm{Cl}_{n+1}^+ \\ & & e_i & \longmapsto & e_i e_{n+1} \end{array}$$

since both algebras have the same dimension and the definition as a morphism yields a one to one correspondence among the basis elements:

$$\Phi(e_{i_1}\cdots e_{i_r}) = e_{i_1}\cdots e_{i_r}(-1)^{\sum_{j=0}^{r-1} j} e_{n+1}^r = \begin{cases} e_{i_1}\cdots e_{i_r} \text{ if } r \text{ is even.} \\ e_{i_1}\cdots e_{i_r} e_{n+1} \text{ if } r \text{ is odd..} \end{cases}$$

This also proves that Φ *is well defined.*

The structure on Cl_n that we will use is the one given by both α and conjugating. These two mappings work remarkably well together.

Lemma 2. *For every* $u \in \mathrm{Cl}_n$*, the identity* $\alpha(\overline{u}) = \overline{\alpha(u)}$ *holds.*

Proof. It is enough to see that it holds for an element of the basis, say $e_{i_1}\cdots e_{i_r}$ with $1 \le i_1 < \cdots < i_r \le n$ and $0 \le r \le n$:

$$\alpha(\overline{e_{i_1}\cdots e_{i_r}}) = \alpha((-1)^r e_{i_r}\cdots e_{i_1}) = (-1)^{2k} e_{i_r}\cdots e_{i_1} = \overline{(-1)^k e_{i_1}\cdots e_{i_r}} = \overline{\alpha(e_{i_1}\cdots e_{i_r})}.$$

□

Observation 5. *The composition* $\overline{\alpha} = \alpha \circ \overline{(\ )} = \overline{(\ )} \circ \alpha$*, which holds because of Lemma 2, is another anti-homomorphism:*

$$\overline{\alpha}(xy) = \overline{\alpha(xy)} = \overline{\alpha(x)\alpha(y)} = \overline{\alpha(y)}\,\overline{\alpha(x)}.$$

Moreover, we have that $\mathrm{Id}_{\mathrm{Cl}_n}^2 = \alpha^2 = \overline{(\ )}^2 = \overline{\alpha}^2 = \mathrm{Id}_{\mathrm{Cl}_n}$*. Thus, the elements* $\mathrm{Id}_{\mathrm{Cl}_n}$*,* α*,* $\overline{(\ )}$ *and* $\overline{\alpha}$ *form a non cyclic finite group of order* 4.

In order to give more structure to Cl_n, we finally provide it with an inner product $\cdot$ and a norm $|\ \ |$.

Definition 22. *We define the* inner product *over the basis of* Cl_n *as the mapping:*

$$\begin{array}{rccc} \cdot & : & \mathrm{Cl}_n \times \mathrm{Cl}_n & \longrightarrow & \mathbb{R} \\ & & (e_{i_1}\cdots e_{i_r}, e_{j_1}\cdots e_{j_s}) & \longmapsto & \begin{cases} 1 \text{ if } r = s, i_k = j_k \text{ for all } k = 1,\ldots,r \\ 0 \text{ otherwise} \end{cases} \end{array},$$

where $1 \leq r, s \leq n$, $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq j_1 < \cdots < j_s \leq n$ *and satisfying linearity.*

We define a norm *on* Cl_n *as the mapping:*

$$\begin{array}{cccc} | \; | : & \mathrm{Cl}_n & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \sqrt{u \cdot u} \end{array}$$

Because of the definition, it is immediately obvious that $\cdot$ is an inner product, and since it does not act over scalars, the coefficients are multiplied amongst them as usually.

Proposition 6. *The mapping* $| \; | : \mathrm{Cl}_n \longrightarrow \mathbb{R}$ *is in fact a norm.*

Proof. For every $u, v \in \mathrm{Cl}_n$ and $\lambda, \mu \in \mathbb{R}$ we have to check:

1. Clearly $|u+v|^2 = (u+v) \cdot (u+v) = u \cdot u + u \cdot v + v \cdot u + v \cdot v \leq u \cdot u + v \cdot v = |u|^2 + |v|^2$, meaning that $|u+v| \leq \sqrt{|u|^2 + |v|^2} \leq |u| + |v|$ since $\lambda + \mu \leq \lambda + \mu + 2\sqrt{\lambda\mu}$.
2. Clearly $|\lambda u| = \sqrt{\lambda u \cdot \lambda u} = \sqrt{\lambda^2 (u \cdot u)} = |\lambda| \, |u|$.
3. We have $0 = |u| = \sqrt{u \cdot u}$ if and oly if $0 = u \cdot u$, that is, the inner product of any element of the basis that expands u with itself is zero, and thus $u = 0$.

And $| \; |$ is a norm. $\square$

In analogous fashion to the complex numbers and the quaternions, we have a real and a complex part of an element of a Clifford algebra.

Definition 23. *We define the* real part *of an element* $u \in \mathrm{Cl}_n$*, and denote it as* $\Re(u)$*, to be the coefficient of* 1 *when* u *is expanded as an* $\mathbb{R}$ *linear combination of the basis monomials* $e_{i_1} \cdots e_{i_r}$*, where* $1 \leq r \leq n$, $1 \leq i_1 < \cdots < i_r \leq n$.

Alternatively, we may consider defining the inner product $\cdot$ as in the complex case. We must first check that this characterization is well defined, that is, that they are equivalent.

Proposition 7. *For every* $u, v \in \mathrm{Cl}_n$*, the following equality holds:*

$$u \cdot v = \Re(\overline{u}v + \overline{v}u)/2.$$

Proof. We first expand u and v in term of the basis elements of Cl_n:

$$u = \sum_{i_1, \ldots, i_r} \lambda_{i_1, \ldots, i_r} e_{i_1} \cdots e_{i_r}, \quad v = \sum_{j_1, \cdots, j_s} \mu_{j_1, \ldots, j_s} e_{j_1} \cdots e_{j_s},$$

with $\lambda_{i_1, \ldots, i_r}, \mu_{j_1, \ldots, j_s} \in \mathbb{R}$ for all $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq j_1 < \cdots < j_s \leq n$.

For the left hand side we can readily check that:

$$u \cdot v = \sum_{i_1, \ldots, i_r} \lambda_{i_1, \ldots, i_r} \mu_{i_1, \ldots, i_r},$$

for indexes $i_1, \dots, i_r$ that both u and v have, since by definition of inner product we must have $r = s$ and $e_{i_k} = e_{j_k}$ for the basis elements to result in non zero numbers.

For the right hand side, we conjugate and obtain:

$$\overline{u} = \sum_{i_1,\dots,i_r} (-1)^r \lambda_{i_1,\dots,i_r} e_{i_r} \cdots e_{i_i}, \quad \overline{v} = \sum_{j_1,\dots,j_s} (-1)^s \mu_{j_1,\dots,j_s} e_{j_s} \cdots e_{j_1}.$$

Because of the definition of product inside Cl_n, in order for the multiplication of basis elements to be a real number, we must have the same number of generators ($r = s$) and (since the basis elements have the generators in a predetermined order) the same generators in the same position ($e_{i_k} = e_{j_k}$). In this case:

$$e_{i_r} \cdots e_{i_i} e_{j_1} \cdots e_{j_s} = e_{i_r} \cdots e_{i_i} e_{i_1} \cdots e_{i_r} = (-1)^r \in \mathbb{R}.$$

Thus, by multiplying as elements of Cl_n:

$$\Re(\overline{u}v) = \sum_{i_1,\dots,i_r} \lambda_{i_1,\dots,i_r} \mu_{i_1,\dots,i_r}, \quad \Re(\overline{v}u) = \sum_{i_1,\dots,i_r} \mu_{i_1,\dots,i_r} \lambda_{i_1,\dots,i_r},$$

for indexes $i_1, \dots, i_r$ that both u and v have, and as the real part of a sum is the sum of real parts:

$$\Re(\overline{u}v + \overline{v}u)/2 = \Re(\overline{u}v)/2 + \Re(\overline{v}u)/2 = \sum_{i_1,\dots,i_r} \lambda_{i_1,\dots,i_r} \mu_{i_1,\dots,i_r},$$

obtaining the desired equality. □

The norm $|\ |$ gives rise to a metric on Cl_n which makes the group of units $\mathrm{Cl}_n^\times$ into a topological group. In particular, $\mathrm{Cl}_n^\times$ is a matrix group itself.

In the following part of the section, we will present the **Clifford groups**, which are subgroups of the group of units Cl_n.

Observation 6. *Via the injective linear transformation $j_n : \mathbb{R}^n \to \mathrm{Cl}_n$ we can identify $\mathbb{R}^n$ with a subspace of Cl_n:*

$$\sum_{r=1}^n x_r \mathbf{e}_r \longleftrightarrow \sum_{r=1}^n x_r e_r$$

which allow us to write elements $x = \mathbf{x} \in \mathbb{R}^n$.

Since $\mathbb{R}^n \subseteq \mathrm{Cl}_n^-$, we have that for every $x \in \mathbb{R}^n$, $u \in \mathrm{Cl}_n^+$ and $v \in \mathrm{Cl}_n^-$:

$$xu, ux \in \mathrm{Cl}_n^-, \quad xv, vx \in \mathrm{Cl}_n^+.$$

Definition 24. *Given $n \geq 1$, we define the* Clifford group Γ_n *as the subgroup:*

$$\Gamma_n = \{u \in \mathrm{Cl}_n^\times : \alpha(u)xu^{-1} \in \mathbb{R}^n \textit{ for all } x \in \mathbb{R}^n\}.$$

This is obviously a group, by simply expanding and using that $\mathbb{R}^n$ is a vector space.

Proposition 8. *We have that* $\mathbb{R}^\times$ *is the centre of* Γ_n*, in particular* $\mathbb{R}^\times \leq \Gamma_n$ *is a normal subgroup.*

Proof. Clearly $\mathbb{R}^\times$ is a normal subgroup as we are inside an $\mathbb{R}$ vector space.

We want to prove that $Z(\Gamma_n) = \mathbb{R}^\times$. Obviously $Z(\Gamma_n) \supseteq \mathbb{R}^\times$ holds. To see $Z(\Gamma_n) \subseteq \mathbb{R}^\times$, we first prove that if $x \in \mathbb{R}^n$, then $x \in \Gamma_n$. As $x(-x)/|x|^2 = |x|^2/|x|^2 = 1$, then $x^{-1} = -x/|x|^2$. Take $y \in \mathbb{R}^n$, we have:

$$\alpha(x)yx^{-1} = (-x)y(-x)/|x|^2 = -yxx/|x|^2 = (-y)(-|x|^2)/|x|^2 = y \in \mathbb{R}^n.$$

Take now $z \in Z(\Gamma_n)$. Since $e_i \in \Gamma_n$ for every $i \in \{1, \dots, n\}$, we have $ze_i = e_iz$. This implies that for every basis monomials $e_{i_1} \cdots e_{i_r}$, where $1 \leq r \leq n$, $1 \leq i_1 < \cdots < i_r \leq n$, we have:

$$ze_{i_1}e_{i_2} \cdots e_{i_r} = e_{i_1}ze_{i_2} \cdots e_{i_r} = \cdots = e_{i_1} \cdots e_{i_r}z,$$

or equivalently z commutes with every element $u \in \mathrm{Cl}_n$. The only invertible elements in Cl_n that commute with every other element are those of $\mathbb{R}^\times$. □

Proposition 9. *The group* Γ_n *is a closed subgroup of* $\mathrm{Cl}_n^\times$

Proof. There is a continuous action:

$$\begin{array}{ccc} \mathrm{Cl}_n^\times \times \mathrm{Cl}_n & \longrightarrow & \mathrm{Cl}_n \\ (u, v) & \longmapsto & \alpha(u)vu^{-1} \end{array}$$

and since $u \in \mathrm{Cl}_n^\times$ implies $\alpha(u) \in \mathrm{Cl}_n^\times$, for each such u the function:

$$\begin{array}{cccc} \rho_u & : & \mathrm{Cl}_n & \longrightarrow & \mathrm{Cl}_n \\ & & v & \longmapsto & \alpha(u)vu^{-1} \end{array}$$

is a linear isomorphism. As $\mathbb{R}^n \subseteq \mathrm{Cl}_n$ is a finite dimensional normed $\mathbb{R}$ subspace, it is closed and we have that the stabiliser:

$$\bigcap_{x \in \mathbb{R}^n} \rho_x^{-1}(x) = \mathrm{Stab}_{\mathrm{Cl}_n^\times}(\mathbb{R}^n) = \{u \in \mathrm{Cl}_n^\times : \rho_x(u) = x \text{ for all } x \in \mathbb{R}^n\} \leq \mathrm{Cl}_n^\times$$

is a closed subgroup. As we have $\mathrm{Stab}_{\mathrm{Cl}_n^\times}(\mathbb{R}^n) = \Gamma_n$, we obtain the desired result. □

Corollary 2. *Given* $n \in \mathbb{N} \setminus \{0\}$*, the group* Γ_n *is a matrix group.*

Proof. It is immediate since Cl_n is a matrix group. □

We wish to find an inclusion of Γ_n into one of the classical matrix groups. We will now prove a couple of technical results needed to progress in the right direction.

Proposition 10. *For* $u \in \Gamma_n$*, we also have* $\alpha(u), \overline{u} \in \Gamma_n$*.*

Proof. Let $u \in \Gamma_n$ and have Observation 5 present. Before anything, note that $\mathbb{R}^n$ is obviously closed under α and $\overline{(\ )}$. To see that it is closed under $\rho_{\alpha(u^{-1})}$, it is enough to see that $u^{-1} \in \Gamma_n$, but this is due to ρ_u being an isomorphism; for every $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x = \alpha(u)yu$, and thus:

$$\alpha(u^{-1})xu^{-1} = \alpha(u^{-1})\alpha(u)yuu^{-1} = y \in \mathbb{R}^n,$$

where we have used the fact that $\alpha(u^{-1})\alpha(u) = 1 = \alpha(u)\alpha(u^{-1})$. Moreover, for $x \in \mathbb{R}^n$ we have $\alpha(x) = -x = \overline{x} \in \mathbb{R}^n$. We first check that $\alpha(u) \in \Gamma_n$:

$$\alpha(\alpha(u))x\alpha(u)^{-1} = \alpha(\alpha(u))\alpha(-x)\alpha(u)^{-1} = \alpha(\alpha(u)(-x)u^{-1}) \in \mathbb{R}^n,$$

We now check that $\overline{u} \in \Gamma_n$:

$$\alpha(\overline{u})x\overline{u}^{-1} = \overline{\alpha(u)}\ \overline{(-x)}\ \overline{u^{-1}} = \overline{u^{-1}(-x)\alpha(u)} = \overline{\alpha(\alpha(u^{-1})\alpha(-x)u)} \in \mathbb{R}^n,$$

where we have also used $\overline{u}\ \overline{u^{-1}} = 1 = \overline{u^{-1}}\ \overline{u}$. □

For $u \in \Gamma_n$ we have that ρ_u keeps $\mathbb{R}^n$ invariant by definition, and thus we can restrict to them the function and not only obtain an isomorphism, but a little more.

Proposition 11. *For each $u \in \Gamma_n$, the function ρ_u restricted to $\mathbb{R}^n$ is an $\mathbb{R}$ linear isometry.*

Proof. For any $x \in \mathbb{R}^n$, we have:

$$|\rho_u(x)|^2 = \left|\alpha(u)xu^{-1}\right|^2 = |\alpha(u)|^2 |x|^2 \left|u^{-1}\right|^2 = |u|^2 |x|^2 |u|^{-2} = |x|^2,$$

hence $|\rho_u(x)| = |x|$. We have used that $\left|u^{-1}\right| = |u|^{-1}$ and $|\alpha(u)| = |u|$, the first being obvious and the second due to Proposition 7 and Lemma 2:

$$|\alpha(u)|^2 = \alpha(u)\overline{\alpha(u)} = \alpha(u)\alpha(\overline{u}) = \alpha(u\overline{u}) = \alpha\left(|u|^2\right) = |u|^2.$$

□

For $u \in \Gamma_n$, this means that if we express ρ_u in terms of the standard basis $\{e_1, \ldots, e_n\}$ we then have $\rho_u \in \mathrm{O}(n)$ and there is a group homomorphism:

$$\begin{array}{cccc} \rho & : & \Gamma_n & \longrightarrow & \mathrm{O}(n) \\ & & u & \longmapsto & \rho_u \end{array}$$

which in fact is also continuous, as it is a linear operator between finite dimensional vector spaces. We are obviously very interested in knowing how this homomorphism behaves.

Proposition 12. *The kernel of ρ is identified as* $\ker \rho = \mathbb{R}^\times = \{t1 : t \in \mathbb{R} \setminus \{0\}\}$.

Proof. Take $u \in \ker \rho$. Decompose $u = u^+ + u^-$ with $u^\pm \in \mathrm{Cl}_n^\pm$, and note that for each $i = 1, \ldots, n$ we may (always) write:

$$u^+ = a_i^+ + e_i b_i^- \text{ and } u^- = a_i^- + e_i b_i^+,$$

with $a_r^\pm, b_r^\pm \in \mathrm{Cl}_n^\pm$ and not involving e_i in their expansions in terms of the basis of $\mathrm{Cl}_n^\pm$, and we have the identities:

$$a_i^+ e_i = e_i a_i^+ \text{ and } -b_i^- e_i = e_i b_i^-.$$

By definition of u, for every $x \in \mathbb{R}^n$ we have $\alpha(u)xu^{-1} = x$, that is, $\alpha(u)x = xu$. With the decomposition we have that:

$$u^+ x = xu^+ \text{ and } -u^- x = xu^-.$$

In particular by setting $x = e_i$ and taking into account the above we obtain:

$$\begin{aligned} a_i^+ e_i + b_i^- &= u^+ e_i = e_i u^+ = a_i^+ e_i - b_i^-, \\ -a_i^- e_i - b_i^+ &= e_i u^- = -u^- e_i = -a_i^- e_i + b_i^+, \end{aligned}$$

which by comparing the parts involving or not involving e_i we see that $b_i^- = 0 = b_i^+$ respectively and thus $u^+ = a_i^+$, $u^- = a_i^-$, neither of these involving e_i. As this is true for all $i = 1, \ldots, n$, we must have $u^+ = t1$ for some $t \in \mathbb{R}$ and $u^- = 0$, hence $u = t1$.

Obviously $\rho_t(x) = x$ if $t \in \mathbb{R}^\times$. □

This knowledge can be used to prove some more technical facts.

Proposition 13. *For every $u \in \Gamma_n$, we have $u\overline{u} \in \mathbb{R}^\times$ and $\overline{u}u = u\overline{u}$. If $v \in \Gamma_n$, we have $uv\overline{uv} = u\overline{u}v\overline{v}$.*

Proof. Take $u \in \Gamma_n$. By Proposition 10 we have that $u\overline{u} \in \Gamma_n$. Taking into account Observation 5 and that we have $\alpha(u^{-1})xu \in \mathbb{R}$, the next shows that $u\overline{u} \in \ker \rho$:

$$\begin{aligned} \rho_{u\overline{u}}(x) &= \alpha(u\overline{u})x(u\overline{u})^{-1} = \alpha(u)\alpha(\overline{u})x\overline{u^{-1}}u^{-1} = \alpha(u)\alpha(\overline{u}\,\overline{x}\alpha(\overline{u^{-1}}))u^{-1} \\ &= \alpha(u)\alpha(\overline{u}\,\overline{x\alpha(u^{-1})})u^{-1} = \alpha(u)\alpha(\overline{\alpha(u^{-1})xu})u^{-1} = \alpha(u)\alpha(-\alpha(u^{-1})xu)u^{-1} \\ &= \alpha(u)\alpha(u^{-1})xuu^{-1} = \alpha(uu^{-1})x(uu^{-1}) = x, \end{aligned}$$

and thus $u\overline{u} \in \mathbb{R}^\times$ by Proposition 12. This means that it commutes with every element of Cl_n, so that:

$$\overline{u}u = u^{-1}u\overline{u}u = u\overline{u}u^{-1}u = u\overline{u}.$$

In fact, if $v \in \Gamma_n$, then $v\overline{v} \in \mathbb{R}^\times$ also commutes with every element of Cl_n and thus:

$$uv\overline{uv} = uv\overline{v}\,\overline{u} = u\overline{u}v\overline{v}.$$

□

Proposition 14. *The function:*

$$\begin{array}{rccc} \nu & : & \Gamma_n & \longrightarrow & \mathbb{R}^\times \\ & & u & \longmapsto & u\overline{u} \end{array} \tag{2}$$

is a continuous group homomorphism.

Proof. In virtue of the first two results of Proposition 13, for every $u \in \Gamma_n$ we have that $\overline{u}u = u\overline{u} \in \mathbb{R}^\times$ and thus:

$$\nu(u) = u\overline{u} = \Re(\overline{u}u + u\overline{u})/2 = |u|^2 \,,$$

which by Proposition 6 is continuous and always takes positive values. Using the last result of Proposition 13, we have that ν is a group homomorphism. □

This last result shows how studying ν's kernel we may obtain a group in which every element behave as a unit and may still be thought via ρ as an element of the orthogonal group. We wish to obtain a morphism relation between those groups.

4 The $Spin(n)$ Group and Representations

4.1 The $Spin(n)$ Group

In this section we define the **spinor groups Spin(n)**, which include our main object of interest. These groups are compact, connected and we will find a surjective homomorphism onto the orthogonal group, as desired.

Definition 25. *Given $n \geq 1$, and $\nu : \Gamma_n \to \mathbb{R}^\times$ as in Equation (2), we define the* pinor group $\mathrm{Pin}(n)$ *as:*

$$\mathrm{Pin}(n) = \ker \nu.$$

We obviously have that $\mathrm{Pin}(n)$ is a closed subgroup of Γ_n, and with respect to the metric induced by the norm $|\ |$ it is also bounded, and hence is compact.

Definition 26. *Given $n \geq 1$, we define the* spinor group $\mathrm{Spin}(n)$ *as:*

$$\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cl}_n^+.$$

Observation 7. *The restriction $\alpha : \mathrm{Pin}(n) \to \mathrm{Pin}(n)$ is a continuous group homomorphism, since it is well defined; take $u \in \mathrm{Pin}(n)$, then:*

$$\nu(\alpha(u)) = |\alpha(u)|^2 = \left|u^+ - u^-\right|^2 = u^+ \cdot u^+ + u^- \cdot u^- = |u|^2 = \nu(u) = 1,$$

and it already is an homomorphism. For this restriction, we have:

$$\mathrm{Spin}(n) = \{u \in \mathrm{Pin}(n) : \alpha(u) = u\} = \mathrm{Pin}(n) \cap \mathrm{Cl}_n^+ \leq \mathrm{Pin}(n),$$

and hence $\mathrm{Spin}(n)$ is a closed subgroup. As we will see in Theorem 4, it is also a normal subgroup.

The natural inclusion $\mathrm{Cl}_n \hookrightarrow \mathrm{Cl}_{n+1}$ yields a natural inclusion $\mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}(n+1)$.

We wish to accomplish the goal set at the end of Section 3, that is, show that the restricted homomorphism $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ is surjective and that $\mathrm{Spin}(n) = \rho^{-1}\mathrm{SO}(n)$. This will be done by showing that $\mathrm{Pin}(n)$ is generated by a set of elements $u \in \mathbb{R}^n$ for which ρ_u is a reflection on a hyperplane. The reason why this is useful will be clear in Proposition 17 and Observation 11.

Observation 8. *The unit sphere $\mathbb{S}^{n-1}$ that lies within $\mathbb{R}^n \subseteq \mathrm{Cl}_n$ has an active role in the following pages. We should observe in particular that it lies within the Clifford algebra and has a simple characterization:*

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} = \left\{ \sum_{r=1}^{n} x_r e_r : \sum_{r=1}^{n} x_r^2 = 1 \right\}.$$

Lemma 3. *Let $u \in \mathbb{S}^{n-1}$. Then $u \in \mathrm{Cl}_n^\times$, it is a unit, and $u^{-1} \in \mathbb{S}^{n-1}$.*

Proof. Take $u \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$, we have:

$$(-u)u = u(-u) = -u^2 = -(-|u|^2) = 1,$$

hence $u^{-1} = -u \in \mathbb{S}^{n-1}$. $\square$

This may be immediately generalized.

Observation 9. *Let $u_1, \dots, u_k \in \mathbb{S}^{n-1}$, by Lemma 3 applied k times we have:*

$$(u_1 \cdots u_k)^{-1} = (-1)^k u_k \cdots u_1 = \overline{u_k \cdots u_1}.$$

This should remind us of the concept of subgroup generated by a set. *In a group G and given a subset $S \subseteq G$, the subgroup generated by S is the smallest subgroup of G containing S, and is denoted by $\langle S \rangle \leq G$. With multiplicative notation, any element $s \in \langle S \rangle$ may be written as a product of elements in S and their inverses:*

$$s = s_1^{\pm 1} \cdots s_k^{\pm 1}.$$

The well known fact that if $H \leq G$ and $S \subseteq H$, then $\langle S \rangle \leq H$ is also useful.

This means that, in fact, Lemma 3 assures that every element of $\left\langle \mathbb{S}^{n-1} \right\rangle$ is a product of elements of $\mathbb{S}^{n-1}$.

Lemma 4. *Let $u, v \in \mathbb{R}^n \subseteq \mathrm{Cl}_n$. If $u \cdot v = 0$, then $vu = -uv$.*

Proof. For $u, v \in \mathbb{R}^n$, we write $u = \sum_{r=1}^n x_r e_r$ and $v = \sum_{r=1}^n y_r e_r$ with $x_r, y_r \in \mathbb{R}^n$ for every $r = 1, \dots, n$. Expanding the multiplication:

$$\begin{aligned}
vu &= \sum_{s=1}^n y_s e_s \sum_{r=1}^n x_r e_r = \sum_{s=1}^n \sum_{r=1}^n y_s x_r e_s e_r = \sum_{r=1}^n y_r x_r e_r^2 + \sum_{r<s} (y_r x_s - y_s x_r) e_r e_s \\
&= -\sum_{r=1}^n x_r y_r - \sum_{r<s} (y_s x_r - y_r x_s) e_r e_s = -u \cdot v - \sum_{r<s} (y_s x_r - y_r x_s) e_r e_s \\
&= -\sum_{r<s} (y_s x_r - y_r x_s) e_r e_s = v \cdot u - \sum_{r<s} (y_s x_r - y_r x_s) e_r e_s \\
&= \sum_{r=1}^n y_r x_r - \sum_{r<s} (y_s x_r - y_r x_s) e_r e_s = -\sum_{r=1}^n x_r y_r e_r^2 - \sum_{r<s} (x_r y_s - x_s y_r) e_r e_s \\
&= -\sum_{r=1}^n \sum_{s=1}^n x_r y_s e_r e_s = \sum_{r=1}^n x_r e_r \sum_{s=1}^n y_s e_s = -uv,
\end{aligned}$$

as desired. $\square$

Observation 10. *We observe that for every $u \in \mathbb{S}^{n-1}$ and $x \in \mathbb{R}^n$, in virtue of Lemma 3 we have:*

$$\alpha(u) x u^{-1} = (-u) x (-u) = uxu.$$

On the one hand if $u \cdot x = 0$ then Lemma 4 guarantees that:

$$\alpha(u)xu^{-1} = uxu = -u^2x = -(-|u|^2)x = -(-1)x = x.$$

On the other hand, if $x = tu$ for some $t \in \mathbb{R}$, then:

$$\alpha(u)xu^{-1} = tu^3 = -tu,$$

so in particular $\alpha(u)xu^{-1} \in \mathbb{R}^n$ for both cases. However, we already know that this always happens because $u \in \mathbb{R}^n \subseteq \Gamma_n$.

Proposition 15. *It holds $\langle \mathbb{S}^{n-1} \rangle \leq \mathrm{Pin}(n)$, that is, the subgroup of Cl_n generated by $\mathbb{S}^{n-1}$ is contained in $\mathrm{Pin}(n)$.*

Proof. As it is already a subgroup, we just have to see that for any $s \in \langle \mathbb{S}^{n-1} \rangle$ we have $s \in \mathrm{Pin}(n)$, that is, $s \in \Gamma_n$ and $\nu(s) = 1$. The latter is direct from Observation 9, since $\nu(s) = s\bar{s} = ss^{-1} = 1$. The former is an immediate consequence of the decomposition $s = s_1 \cdots s_k$ with $s_r \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n \subseteq \Gamma_n$ for all $r = 1, \ldots, k$ and thus $\alpha(s_r)y(s_r)^{-1} \in \mathbb{R}^n$ for any $y \in \mathbb{R}^n$:

$$\alpha(s_1 \cdots s_k)x(s_1 \cdots s_k)^{-1} = ((-1)^k s_1 \cdots s_k)x((-1)^k s_k \cdots s_1) \in \mathbb{R}^n.$$

□

The next step is to study the restriction of the homomorphism $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ to $\rho : \langle \mathbb{S}^{n-1} \rangle \to \mathrm{O}(n)$. As we have seen, for each $u \in \mathbb{S}^{n-1}$ we have an $\mathbb{R}$ linear isometry:

$$\begin{array}{rccc} \rho_u & : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & x & \longmapsto & uxu. \end{array}$$

More generally, we have also seen that considering $u \in \langle \mathbb{S}^{n-1} \rangle$ with $u = u_1 \cdots u_k$ for $u_1, \ldots, u_k \in \mathbb{S}^{n-1}$, then for any $x \in \mathbb{R}^n$:

$$\rho_u(x) = \alpha(u_1 \cdots u_k)x(u_1 \cdots u_k)^{-1} = u_1 \cdots u_k x u_k \cdots u_1.$$

Proposition 16. *For $u \in \mathbb{S}^{n-1}$, the mapping $\rho_u : \mathbb{R}^n \to \mathbb{R}^n$ is a reflection in the hyperplane orthogonal to u.*

Proof. What we have seen in Observation 10 is precisely that ρ_u meets the defining equation of a hyperplane reflection, and thus it is so with respect to the hyperplane orthogonal to u. □

We can finally show that $\mathbb{S}^{n-1}$ actually generates $\mathrm{Pin}(n)$, and thus the elements of $\mathrm{Spin}(n)$ are relatively easy to work with.

Proposition 17. 1. *The mapping $\rho : \langle \mathbb{S}^{n-1} \rangle \to \mathrm{O}(n)$ is surjective with $\ker \rho = \{1, -1\}$.*

2. *The mapping* $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ *is surjective with* $\ker \rho = \{1, -1\}$.

3. *It holds* $\langle \mathbb{S}^{n-1} \rangle = \mathrm{Pin}(n)$.

Proof. 1. Computing $\ker \rho = \{u \in \langle \mathbb{S}^{n-1} \rangle : \rho_u = \mathrm{Id}_{\mathrm{O}(n)}\}$, we must have $uxu = x$ for all $x \in \mathbb{R}^n$, which only happens for $u = \pm 1$. Because of Observation 10, any reflection in the hyperplane orthogonal to $u \in \mathbb{S}^{n-1}$ has the form ρ_u. The surjectivity is due since every element $A \in \mathrm{O}(n)$ is a product of hyperplane reflections, as stated by Proposition 2.1.

2. The surjectivity follows from the above, since $\langle \mathbb{S}^{n-1} \rangle \subseteq \mathrm{Pin}(n)$. We obviously have $\pm 1 \in \ker \rho$. Let $t \in \ker \rho \subseteq \mathrm{Pin}(n) \cap \mathbb{R}^\times$, then $1 = \nu(t) = |t|^2 = t\bar{t} = t^2$, so $t = \pm 1$.

3. We just have to see that $\mathrm{Pin}(n) \subseteq \langle \mathbb{S}^{n-1} \rangle$. Take $v \in \mathrm{Pin}(n)$, then $\rho_v \in \mathrm{O}(n)$ can be expressed as a product of hyperplane reflections as noted above, each of which has the form ρ_u for some vector $u \in \mathbb{S}^{n-1}$. Hence $\rho_v = \rho_{u_1 \cdots u_k}$ for some $u_1, \ldots, u_k \in \mathbb{R}^{n-1}$. But then:

$$\mathrm{Id}_{\mathrm{O}(n)} = \rho_v \rho_v = \rho_v \rho_{u_1 \cdots u_k} = \rho_{v u_1 \cdots u_k} \Longrightarrow v u_1 \cdots u_k \in \ker \rho = \{\pm 1\},$$

hence $v = \pm (u_1 \cdots u_k)^{-1} = \pm (-1)^k u_k \cdots u_1 \in \langle \mathbb{S}^{n-1} \rangle$.

□

Theorem 4. *1. The group* $\mathrm{Pin}(n)$ *is the disjoint union of open subsets:*

$$\mathrm{Pin}(n) = \left(\mathrm{Pin}(n) \cap \mathrm{Cl}_n^+\right) \cup \left(\mathrm{Pin}(n) \cap \mathrm{Cl}_n^-\right) = \mathrm{Spin}(n) \cup \left(\mathrm{Pin}(n) \cap \mathrm{Cl}_n^-\right).$$

2. *The group* $\mathrm{Spin}(n)$ *is a normal subgroup of* $\mathrm{Pin}(n)$ *and every element* $u \in \mathrm{Spin}(n)$ *can be expressed as a product of even length of elements of* $\mathbb{S}^{n-1}$*, that is,* $u = u_1 \cdots u_{2k}$ *with* $u_1, \ldots, u_{2k} \in \mathbb{S}^{n-1}$.

3. *For any* $v \in \mathbb{S}^{n-1}$ *it holds* $\mathrm{Pin}(n) \cap \mathrm{Cl}_n^- = v\mathrm{Spin}(n)$*, and every element* $u \in \mathrm{Pin}(n) \cap \mathrm{Cl}_n^-$ *can be expressed as a product of odd length of elements of* $\mathbb{S}^{n-1}$*, that is,* $u = u_1 \cdots u_{2k+1}$ *with* $u_1, \ldots, u_{2k+1} \in \mathbb{S}^{n-1}$.

Proof. 1. We have $\mathrm{Pin}(n) = \langle \mathbb{S}^{n-1} \rangle$ and thus any element $u \in \mathrm{Pin}(n)$ may be expressed as $u = u_1 \cdots u_k$ for some $u_1, \ldots, u_k \in \mathbb{S}^{n-1}$. For this decomposition we have $\alpha(u_1 \cdots u_k) = (-1)^k u_1 \cdots u_k$, meaning that $u \in \mathrm{Pin}(n) \cap \mathrm{Cl}_n^{\pm}$ depending on k being even or odd respectively. This proves the equality.

The subsets are disjoint since $\mathrm{Cl}_n^+ \cap \mathrm{Cl}_n^- = \{0\}$ but $0 \notin \mathrm{Pin}(n)$.

2. Take $u \in \mathrm{Spin}(n)$ and $v \in \mathrm{Pin}(n)$. We know by the above that $u = u_1 \cdots u_{2k}$ for some $u_1, \ldots, u_{2k} \in \mathbb{S}^{n-1}$. If $v \in \mathrm{Spin}(n)$, then obviously $vuv^{-1} \in \mathrm{Spin}(n)$. Suppose $v = v_1 \cdots v_{2l+1}$ with $v_1, \ldots, v_{2l+1} \in \mathbb{S}^{n-1}$, now:

$$vuv^{-1} = -v_1 \cdots v_{2l+1} u_1 \cdots u_{2k} v_{2l+1} \cdots v_1 = w_1 \cdots w_{2(2l+1+k)} \in \mathrm{Spin}(n),$$

with $w_1, \ldots, w_{2(2l+1+k)} \in \mathbb{S}^{n-1}$ by renaming the elements, and there is an even number of them. This proves the normality of the group.

3. Take $u \in \mathrm{Pin}(n) \cap \mathrm{Cl}_n^-$. We know by the above that $u = u_1 \cdots u_{2k+1}$ for some $u_1, \dots, u_{2k+1} \in \mathbb{S}^{n-1}$. Moreover, for each $v \in \mathbb{S}^{n-1}$ we have that:

$$v^{-1}u = (-v)u \in \mathrm{Pin}(n) \cap \mathrm{Cl}_n^+,$$

say $v^{-1}u = w$, and so $u = vw$ with $w \in \mathrm{Spin}(n)$. This argument can be applied at the inverse, as given $w \in \mathrm{Spin}(n)$ we have that:

$$vw \in \mathrm{Pin}(n) \cap \mathrm{Cl}_n^-,$$

say $vw = u$ with $u \in \mathrm{Pin}(n) \cap \mathrm{Cl}_n^-$. We thus have both inclusions and the equality. □

Observation 11. *The following notation is extensively used in the bibliography:*

$$\mathrm{Pin}(n)^+ = \mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cl}_n^+, \quad \mathrm{Pin}(n)^- = \mathrm{Pin}(n) \cap \mathrm{Cl}_n^-.$$

It is also worth noting that these two groups can also be characterized using the surjective homomorphism $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$. For $u_1, \dots, u_k \in \mathbb{S}^{n-1}$ we have that ρ_{u_i} is a hyperplane reflection for every $i = 1, \dots, k$, for which $\det \rho_{u_i} = -1$ as guaranteed by Lemma 1. Thus:

$$\det \rho_{u_1 \cdots u_k} = \det \rho_{u_1} \cdots \rho_{u_k} = \det \rho_{u_1} \cdots \det \rho_{u_k} = (-1)^k.$$

Now recalling Definition 5 together with Theorem 4 we have that for $u \in \mathrm{Pin}(n)$:

$$\begin{aligned} u \in \mathrm{Spin}(n) &\iff \rho_u \in \mathrm{SO}(n), \\ u \in \mathrm{Pin}(n)^- &\iff \rho_u \in \mathrm{O}(n)^-. \end{aligned}$$

Theorem 5. *The continuous homomorphism $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ is surjective and:*

$$\rho^{-1}(\mathrm{SO}(n)) = \mathrm{Spin}(n), \quad \rho^{-1}(\mathrm{O}(n)^-) = \mathrm{Pin}(n)^-.$$

Meaning that the restriction $\rho^+ = \rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is also surjective and the kernels of these homomorphisms are $\ker \rho = \ker \rho^+ = \{\pm 1\}$.

Proof. We know that $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ is surjective by Proposition 17. We obtain the two equalities in virtue of Observation 11, and since $\pm 1 \in \mathrm{Spin}(n)$, the kernel is maintained. □

As we wished, Theorem 5 guarantees that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$.

Observation 12. *This characterization leads to an even better understanding of the group by means of topological considerations. In particular, as is shown in [13], it can be used to prove that the center of $\mathrm{Spin}(n)$ is:*

$$Z(\mathrm{Spin}(n)) = \begin{cases} \{\pm 1\} \cong \mathbb{Z}_2 \text{ if } n \text{ is odd.} \\ \{\pm 1, \pm e_1 \cdots e_n\} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ if } n \equiv 0 \bmod 4. \\ \mathbb{Z}_4 \text{ if } n \equiv 2 \bmod 4. \end{cases} \end{cases}$$

It also provides a way of defining a (non injective) product $\mathrm{Spin}(p) \times \mathrm{Spin}(q) \to \mathrm{Spin}(p+q)$ *that covers the product:*

$$\begin{array}{rccc} \oplus & : & \mathrm{SO}(p) \times \mathrm{SO}(q) & \longrightarrow & \mathrm{SO}(p+g) \\ & & (A, B) & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{array}$$

Example 4. *In low dimensions, there exist isomorphisms between the Spin groups and other Classical groups. These are not featured in higher dimensions, thus being called accidental isomorphisms. The first* 6 *Spin groups are:*

Spin group	Classical group
Spin(1)	O(1)
Spin(2)	$\mathrm{SO}(2) \cong \mathrm{U}(1)$
Spin(3)	$\mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong \mathbb{S}^3$
Spin(4)	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
Spin(5)	Sp(2)
Spin(6)	SU(4)

But only vestiges of these isomorphisms remain in dimensions 7 *and* 8*, completely disappearing afterwards. The ensuing groups are of its own kind.*

4.2 The *Spin*(n) Representations

Now that we are familiar with the Spin(n) groups, in this section we present some elementary and not so elementary facts concerning their **representations**. As our goal is to study the automorphisms of Spin(n), the results, which can be found in [1], are provided without proofs.

We wish to find a continuous homomorphism $\varphi : \mathrm{Spin}(n) \to \mathrm{M}_m(\mathbb{C})$ that induces a representation of Spin(n). In a first approach, we may consider the projection $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n) \subset \mathrm{GL}_n(\mathbb{C})$, that clearly induces a representation. This representation has $-1 \in \mathrm{Spin}(n)$ acting the same way as $1 \in \mathrm{Spin}(n)$ does, namely as the identity element. In fact, this is a distinguishing factor of λ, since we may differentiate representations in which this happens and in which it does not. In the former, we have that the representation must factorize through SO(n), and we simply have a lift to Spin(n) of a representation in SO(n). Thus, the representation λ is in fact the canonical representation of SO(n) acting naturally over $\mathbb{R}^n$, has dimension n and is irreducible.

We shall now consider representations that do not factorize through SO(n), distinguishing when the dimension is even or odd.

Proposition 18. *Let* $n \in \mathbb{N}$*. If* $n = 2r + 1$ *is odd, then* Spin(n) *has one irreducible representation* Δ *of degree* 2^r*. If* $n = 2r$ *is even, then* Spin(n) *has two irreducible representations* Δ^+*,* Δ^- *of degree* 2^{r-1}*.*

Sketch of the proof. Consider the Spin groups over the complex numbers. If $n = 2r+1$, we may consider the inclusions:

$$\mathrm{Spin}(2r+1) \subset \mathrm{Cl}^+_{2r+1} \cong \mathrm{Cl}_{2r} \hookrightarrow \mathrm{Cl}_{2r} \otimes \mathbb{C} \cong \mathrm{M}_{2^r}(\mathbb{C}).$$

If $n = 2r$, we may consider the inclusions:

$$\mathrm{Spin}(2r) \subset \mathrm{Cl}^+_{2r} \cong \mathrm{Cl}_{2r-1} \hookrightarrow \mathrm{Cl}_{2r-1} \otimes \mathbb{C} \cong \mathrm{M}_{2^{r-1}}(\mathbb{C}) \oplus \mathrm{M}_{2^{r-1}}(\mathbb{C}).$$

The periodicity results in [12] are to be used to see the last equivalences. □

These representations are complex. Notice how a representation on $\mathrm{Spin}(n)$ may be the composition of the inclusion on a higher dimensional group $\mathrm{Spin}(n+r)$ (for some $r \in \mathbb{N}$) with a representation on the latter. We observe that Δ^+ is distinguishable from Δ^- in the sense that by definition they take different values in some central elements:

$$\Delta^+(i^r e_1 \cdots e_{2r}) = \mathrm{Id}_{\mathrm{GL}_{2^{r-1}}(\mathbb{C})},$$
$$\Delta^-(i^r e_1 \cdots e_{2r}) = -\mathrm{Id}_{\mathrm{GL}_{2^{r-1}}(\mathbb{C})},$$

being $i = \sqrt{-1}$ the complex number.

Proposition 19. *For every $r \in \mathbb{N}$, under the inclusions:*

$$\mathrm{Spin}(2r) \longrightarrow \mathrm{Spin}(2r+1) \longrightarrow \mathrm{Spin}(2r+2),$$

we have that:

$$\begin{tikzcd}
& & \Delta^+ \arrow[dl, mapsto] \\
\Delta^+ + \Delta^- & \Delta \arrow[l, mapsto] & \\
& & \Delta^- \arrow[ul, mapsto]
\end{tikzcd}$$

We wish to know when these representations are real, if they can ever be.

Proposition 20. *Let $r \in \mathbb{N}$. The representation Δ of $\mathrm{Spin}(2r+1)$ is real if $2r+1 \equiv 1, 7 \bmod 8$. The representations Δ^+ and Δ^- of $\mathrm{Spin}(2r)$ are real if $2r \equiv 0 \bmod 8$.*

This last result is difficult to prove, but it immediately meets the eye that the case of $\mathrm{Spin}(8)$ must be a special one, since $\dim(\lambda) = \dim(\Delta^+) = \dim(\Delta^-) = 8$. This means in particular that all three irreducible representations are real, and in fact $\Delta^\pm(e_1 \cdots e_8) = \mathrm{Id}_{\mathrm{GL}_8(\mathbb{C})}$.

Theorem 6. *The only irreducible representations of $\mathrm{Spin}(8)$ are λ, Δ^+ and Δ^-.*

This concludes what we will need to know about the $\mathrm{Spin}(n)$ representations in order to be able to sow Theorem 7.

5 *Spin*(8) and Triality

5.1 The Automorphisms of *Spin*(8)

In this section we present the main result that makes Spin(8) special, that is, that its group of **outer automorphisms** is isomorphic to the group of permutations of three elements. Most of the auxiliary results used can be found in [1].

Definition 27. *Let A be an algebra over $\mathbb{K}$. The group of K-*algebra automorphisms *of A is* $\mathrm{Aut}_{\mathbb{K}}(A)$.

In general, this group contains many elements. In particular, for every unit $u \in A^\times$ the conjugation:

$$\begin{array}{cccc} \chi_u & : & A & \longrightarrow & A \\ & & a & \longmapsto & uau^{-1} \end{array}$$

is an automorphism.

Definition 28. *Let A be an algebra over $\mathbb{K}$. An automorphism having the form χ_u for some unit $u \in A^\times$ is called an* inner automorphism, *and form the group of inner automorphisms* $\mathrm{Inn}_{\mathbb{K}}(A)$, *which is a subgroup of* $\mathrm{Aut}_{\mathbb{K}}(A)$.

These inner automorphisms are readily understood, making our interest lie among the ones that are not of this form.

Definition 29. *Let A be an algebra over $\mathbb{K}$. The* group of outer automorphisms *of A is defined as* $\mathrm{Out}_{\mathbb{K}}(A) = \mathrm{Aut}_{\mathbb{K}}(A)/\mathrm{Inn}_{\mathbb{K}}(A)$. *The equivalence classes of* $\mathrm{Out}_{\mathbb{K}}(A)$ *are called* outer automorphisms.

To completely determine the group of outer automorphisms of Spin(8) we need a few preliminary results.

Lemma 5. *Let G be a compact group acting on a vector space $\mathbb{V}$. We can always find a positively defined quadratic form in $\mathbb{V}$ that remains invariant by the action of the group.*

Proof. It can be found in [7]. □

Proposition 21. *Let G be a compact simple Lie group. Then* $\mathrm{End}(G) = \{1\} \cup \mathrm{Aut}(G)$.

Proof. It can be found in [7]. □

The main result that we wish to prove is:

Theorem 7. *The group of outer automorphisms of* Spin(8) *is the group of permutations of three elements:* $\mathrm{Out}_{\mathbb{K}}(\mathrm{Spin}(8)) = \Sigma_3$.

Proof. We will prove that the elements of $\mathrm{Out}_{\mathbb{R}}(\mathrm{Spin}(8))$ are in a one to one correspondence with the permutations of the three representations λ, Δ^+ and Δ^-.

If we have an automorphism $\alpha : \mathrm{Spin}(8) \to \mathrm{Spin}(8)$ and a representation $\rho : \mathrm{Spin}(8) \to \mathrm{GL}_8(\mathbb{C})$, then clearly $\rho \circ \alpha : \mathrm{Spin}(8) \to \mathrm{GL}_8(\mathbb{C})$ is also a representation. Thus, we may define an homomorphism:

$$\begin{array}{cccccccc} \psi & : & \mathrm{Out}_{\mathbb{R}}(\mathrm{Spin}(8)) & \longrightarrow & \Sigma_3\{\lambda, \Delta^+, \Delta^-\} & & & \\ & & \alpha & \longmapsto & \psi(\alpha) & : & \{\lambda, \Delta^+, \Delta^-\} \longrightarrow \{\lambda, \Delta^+, \Delta^-\} & \\ & & & & & & [\rho] \longmapsto [\rho \circ \alpha] & \end{array}$$

It is important to see that we must consider the class of the irreducible representation $[\rho]$, since as in Definition 17 we are taking the representations modulo equivalence. In our case in particular, this equivalence translates as taking representations modulo a change of basis in $\mathrm{GL}_8(\mathbb{C})$. We have to show that ψ is both surjective and injective.

We claim that ψ is surjective. Consider the diagram:

$$\begin{tikzcd}
 & \mathrm{Spin}(8) \arrow[d] \\
 & \mathrm{SO}(8) \arrow[d, hook] \\
 & \mathrm{O}(8) \arrow[d, hook] \\
 & \mathrm{GL}_8(\mathbb{R}) \arrow[d, hook] \\
\mathrm{Spin}(8) \arrow[r, "\Delta^+"] \arrow[ur, dashed] \arrow[uur, dashed] \arrow[uuur, dashed] \arrow[uuuur, dashed, "\alpha^+"] & \mathrm{GL}_8(\mathbb{C})
\end{tikzcd}$$

The first factorization is justified because Δ^+ is a real representation. In virtue of Lemma 5, we may choose an orthonormal basis and thus factorize through $\mathrm{O}(8)$. As $\mathrm{Spin}(8)$ is connected, it must have image into a connected group, allowing the factorization via $\mathrm{SO}(8)$. Finally, since $\mathrm{Spin}(8)$ is simply connected for every $n > 2$, it is the universal cover of $\mathrm{SO}(8)$ and thus we may lift the map to $\alpha^+ : \mathrm{Spin}(8) \to \mathrm{Spin}(8)$.

Taking into account Proposition 21 and the fact that Δ^+ is not trivial, we must have that α^+ is a non trivial automorphism. Observe that by construction $\lambda \circ \alpha^+ = \Delta^+$ and thus the representation λ is in fact the one that makes the diagram above commute. Let ρ be another representation acting after α^+, by definition of ψ we have that:

$$\psi(\alpha^+)(\rho) : \mathrm{Spin}(8) \xrightarrow{\alpha^+} \mathrm{Spin}(8) \xrightarrow{\rho} \mathrm{GL}_8(\mathbb{C}).$$

The diagram above shows that over λ this composition yields Δ^+, but the other two representations are left undetermined. We may have one of the following:

$$\psi(\alpha^+):\begin{cases}\lambda \longmapsto \Delta^+ \\ \Delta^+ \longmapsto \lambda \\ \Delta^- \longmapsto \Delta^-\end{cases} \quad , \quad \psi(\alpha^+):\begin{cases}\lambda \longmapsto \Delta^+ \\ \Delta^+ \longmapsto \Delta^- \\ \Delta^- \longmapsto \lambda\end{cases} \quad .$$

An analogous argument starting with Δ^- obtains a non trivial automorphism α^- that acts on the representations as one of the following:

$$\psi(\alpha^-):\begin{cases}\lambda \longmapsto \Delta^- \\ \Delta^+ \longmapsto \lambda \\ \Delta^- \longmapsto \Delta^+\end{cases} \quad , \quad \psi(\alpha^-):\begin{cases}\lambda \longmapsto \Delta^- \\ \Delta^+ \longmapsto \Delta^+ \\ \Delta^- \longmapsto \lambda\end{cases} \quad .$$

To see that ψ results in all the elements of $\Sigma_3\{\lambda, \Delta^+, \Delta^-\}$, and thus is surjective, we notice that regardless of the respective order of $\psi(\alpha^+)$ and $\psi(\alpha^-)$ they always generate the whole permutation group as we wished. The options are:

$\psi(\alpha^+)$	$\psi(\alpha^-)$	Result
2	3	Two permutations of different order generate Σ_3
3	2	Two permutations of different order generate Σ_3
2	2	Two different permutations of order 2 generate Σ_3

We claim that ψ is injective. Let $\alpha : \mathrm{Spin}(8) \to \mathrm{Spin}(8)$ be an automorphism such that $\psi(\alpha) = \mathrm{Id}_{\Sigma_3}$, namely $\psi(\alpha)(\rho) = \rho$ for every $\rho = \lambda, \Delta^+, \Delta^-$. It is enough to show that α is an inner automorphism to ensure the desired result.

The above means that we have $[\lambda \circ \alpha] = [\lambda]$, they are equivalent representations. Thus, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Spin}(8) \arrow[r, "\alpha"] \arrow[d, "\lambda"] & \mathrm{Spin}(8) \arrow[d, "\lambda"] \\
\mathrm{O}(8) \arrow[r, "\chi_U"] & \mathrm{O}(8)
\end{tikzcd}$$

where χ_U is a change of basis via certain element $U \in \mathrm{O}(8)$. We may expand it onto the following diagram:

$$\begin{tikzcd}
& \mathrm{Spin}(8) \arrow[rrr, "\alpha"] \arrow[dl, "\pi"'] \arrow[dr, hook, "j"] & & & \mathrm{Spin}(8) \arrow[dr, "\pi"] \arrow[dl, hook', "j"'] & \\
\mathrm{SO}(8) \arrow[dr, hook, "i"] & & \mathrm{Pin}(8) \arrow[r, dashed, "\chi_{\tilde{U}}"] \arrow[dl, "\overline{\pi}"] & \mathrm{Pin}(8) \arrow[dr, "\overline{\pi}"] & & \mathrm{SO}(8) \arrow[dl, hook', "i"'] \\
& \mathrm{O}(8) \arrow[rrr, "\chi_U"] & & & \mathrm{O}(8) &
\end{tikzcd}$$

where both the left and right rhombuses and the lower trapezoid are commutative by construction, as we have taken $\tilde{U} = \overline{\pi}^{-1}(U) \in \mathrm{Pin}(8)$ that exists because of $\overline{\pi}$ being surjective, and i, j denote the natural inclusions.

If there is justice in this world, the upper trapezoid should be commutative, that is, $j \circ \alpha = \chi_{\tilde{U}} \circ j$. Let $\beta_1 = j \circ \alpha$, $\beta_2 = \chi_{\tilde{U}} \circ j$. What has been said about the commutativity of the inner diagrams allow us to say:

$$\overline{\pi} \circ \beta_1 = \overline{\pi} \circ j \circ \alpha = i \circ \pi \circ \alpha = \chi_U \circ i \circ \pi = \chi_U \circ \overline{\pi} \circ j = \overline{\pi} \circ \chi_{\tilde{U}} \circ j = \overline{\pi} \circ \beta_2.$$

That is, we have the diagram:

$$\begin{tikzcd} \mathrm{Spin}(8) \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] \arrow[dr, dashed] & \mathrm{Pin}(8) \arrow[d, "\overline{\pi}"] \\ & \mathrm{O}(8) \end{tikzcd}$$

but as $\mathrm{Spin}(8)$ is connected, it must have image into a connected group of $\mathrm{Pin}(8)$, namely itself. So in fact as $i \circ \pi = \overline{\pi} \circ j$, we have:

$$\begin{tikzcd} \mathrm{Spin}(8) \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] \arrow[dr, dashed, "\pi"'] & \mathrm{Spin}(8) \arrow[r, hook, "j"] \arrow[d, "\pi"] & \mathrm{Pin}(8) \arrow[d, "\overline{\pi}"] \\ & \mathrm{SO}(8) \arrow[r, hook, "i"] & \mathrm{O}(8) \end{tikzcd}$$

being β_1 and β_2 morphisms, in particular $\beta_1(1) = 1 = \beta_2(1)$. As we may think these morphisms as a lift of the projection onto $\mathrm{SO}(8)$ that have a point in common, basic covering theory assure that $\beta_1 = \beta_2$, they are equal and the upper trapezoid is commutative.

Moreover, what we have seen is that we do not need to project onto $\mathrm{Pin}(8)$ to get the equality as this holds already in $\mathrm{Spin}(8)$, and thus in fact $\alpha = \chi_{\tilde{U}}$, it is a conjugation by an element $\tilde{U} \in \mathrm{Pin}(8)$. We have that for every element $x \in \mathrm{Spin}(8)$ it holds $\alpha(x) = \tilde{U} x \tilde{U}^{-1}$. It is enough to see that $\tilde{U} \in \mathrm{Spin}(8)$ to obtain that α is a conjugation in the desired way, and thus an inner automorphism.

Suppose that it is not an inner automorphism, or equivalently $\tilde{U} \notin \mathrm{Spin}(8)$. By Theorem 4, we can write $\tilde{U} = e_1\theta$ for some $\theta \in \mathrm{Spin}(8)$. As a conjugation by an element of $\mathrm{Spin}(8)$ leads to equivalent outer automorphisms, we have that $\alpha = \chi_{e_1}$ and must only check how the conjugation acts. However, choosing Δ^+ we have that:

$$\Delta^+ \circ \chi_{e_1}(e_1 \cdots e_8) = \Delta^+(e_1 e_1 \cdots e_8(-e_1)) = \Delta^+(-e_1 \cdots e_8) = -\mathrm{Id}_{\mathrm{GL}_8(\mathbb{R})},$$

that does not fix the representation Δ^+, whereas we had started with $\Delta^+ \circ \alpha = \psi(\alpha)(\Delta^+) = \Delta^+$. As this is a contradiction, we must have that $\tilde{U} \in \mathrm{Spin}(8)$, as we claimed. □

Example 5. *It is enlightening to see how* $\psi : \mathrm{Inn}_{\mathbb{R}}(\mathrm{Spin}(8)) \to \Sigma_3\{\lambda, \Delta^+, \Delta^-\}$*, the analogous homomorphism, works. For any invertible* $u \in \mathrm{Spin}(8)^\times$*, this is simply the* *diagrams below:*

$$\begin{tikzcd} \mathrm{Spin}(8) \arrow[r, "\rho"] \arrow[d, "\chi_u"'] & \mathrm{GL}_8(\mathbb{C}) \arrow[d, "\chi_{\rho(u)}"'] \\ \mathrm{Spin}(8) \arrow[r, "\rho"] & \mathrm{GL}_8(\mathbb{C}) \end{tikzcd} \text{ , \textit{completed as} } \begin{tikzcd} h \arrow[r, "\rho"] \arrow[d, "\chi_u"'] & \rho(h) \arrow[d, "\chi_{\rho(u)}"'] \\ uhu^{-1} \arrow[r, "\rho"] & \rho(u)\rho(h)\rho(u)^{-1} \end{tikzcd}$$

Note how the representations ρ and $\rho \circ \chi_u$ are trivially equivalent.

5.2 The Fundamentals of Triality

In this section we recall the concept of **duality** and define the concept of **triality**.

Definition 30. *Let U, V be two vector spaces over a field $\mathbb{K}$. We say that there is a* duality *between them if there exists a non degenerate linear map:*

$$f : U \otimes V \longrightarrow \mathbb{K}.$$

The non degeneracy is expressed as the two following conditions:

$$\begin{aligned} f(u \otimes v) = 0 \text{ for every } u &\iff v = 0, \\ f(u \otimes v) = 0 \text{ for every } v &\iff u = 0. \end{aligned}$$

Equivalently, we may have a non degenerate bilinear map $f : U \times V \to \mathbb{K}$.

Example 6. *The canonical example of a duality occurs between a $\mathbb{K}$ vector space U and its dual U^*, where the linear function is the so called incidence:*

$$\begin{array}{cccc} j : & U \otimes U^* & \longrightarrow & \mathbb{K} \\ & v \otimes w & \longmapsto & w(v) \end{array}$$

The characterization of a duality over finite dimensional vector spaces is relatively straightforward.

Proposition 22. *Let U, V be two finite dimensional vector spaces over a field $\mathbb{K}$ with a duality $f : U \otimes V \to \mathbb{K}$. It exists an isomorphism $\varphi : V \to U^*$ that makes the following diagram commute:*

$$\begin{tikzcd} U \otimes V \arrow[r, "f"] \arrow[d, "\mathrm{Id}_U \otimes \varphi"'] & \mathbb{K} \\ U \otimes U^* \arrow[ur, "j"'] & \end{tikzcd}$$

Proof. Define:

$$\begin{array}{cccccccc} \varphi & : & V & \longrightarrow & U^* & & & \\ & & v & \longmapsto & \varphi(v) & : & U \longrightarrow & \mathbb{K} \\ & & & & & & u \longmapsto & f(u \otimes v) \end{array}$$

The non degeneracy conditions imply that φ is injective, and thus $\dim(V) \leq \dim(U^*) = \dim(U)$. Interchanging the roles of U and V, we obtain that $\dim(U) \leq \dim(V^*) = \dim(V)$. As so, $\dim(U) = \dim(V)$ and φ is an isomorphism, that makes the diagram commute by definition. $\square$

So in fact a duality is essentially the relation between a vector space and its dual. This relation translates to a matrix level to taking the transposed of the original matrix.

Definition 31. *Let V_1, V_2, V_3 be three finite dimensional vector spaces over a field $\mathbb{K}$. We say that there is a* triality *between them if there exists a non degenerate linear map:*

$$f : V_1 \otimes V_2 \otimes V_3 \longrightarrow \mathbb{K}.$$

The non degeneracy is expressed as the three following conditions:

$$\begin{aligned}
f(v_1 \otimes v_2 \otimes v_3) = 0 \text{ for every } v_1 &\iff v_2 = 0 \text{ or } v_3 = 0, \\
f(v_1 \otimes v_2 \otimes v_3) = 0 \text{ for every } v_2 &\iff v_1 = 0 \text{ or } v_3 = 0, \\
f(v_1 \otimes v_2 \otimes v_3) = 0 \text{ for every } v_3 &\iff v_1 = 0 \text{ or } v_2 = 0.
\end{aligned}$$

In particular, we have a duality for every two pairs from V_1, V_2, V_3.

Example 7.

1. *Consider $K = V_1 = V_2 = V_3 = \mathbb{R}$ with:*

$$\begin{array}{rccc} f & : & \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x \otimes y \otimes z & \longmapsto & xyz \end{array}$$

   This is obviously a triality.

2. *Consider $K = \mathbb{R}$, $V_1 = V_2 = V_3 = \mathbb{C}$ with:*

$$\begin{array}{rccc} f & : & \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} & \longrightarrow & \mathbb{R} \\ & & x \otimes y \otimes z & \longmapsto & \Re(xyz) \end{array}$$

   By letting $x = x_1 + ix_2$, $y = y_1 + iy_2$, $z = z_1 + iz_2$ and doing a simple calculation, we obtain that $\Re(xyz) = x_1y_1z_1 - x_2y_2z_1 - x_2y_1z_2 - x_1y_2z_2$. We will see the property for the first component, the others are analogous. If either $y = 0$ or $z = 0$, clearly this is zero. Conversely, let $f(x \otimes y \otimes z) = 0$ for all $x \in \mathbb{C}$ and without loss of generality take $y \neq 0$ (say $y_1 \neq 0$). By choosing x accordingly, we have:

$$\begin{aligned}
x_1 = 1, x_2 = 0 &\Longrightarrow y_1z_1 = y_2z_2, \\
x_1 = 0, x_2 = 1 &\Longrightarrow y_2z_1 = -y_1z_2,
\end{aligned}$$

   and thus $y_1z_1^2 = y_2z_1z_2 = -y_1z_2^2$, that is, $z_1^2 = -z_2^2$, which is only possible if $z_1 = z_2 = 0$, as we wished. This is then a triality.

3. *Consider $K = \mathbb{R}$, $V_1 = V_2 = V_3 = \mathbb{H}$ with:*

$$\begin{array}{rccc} f & : & \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} & \longrightarrow & \mathbb{R} \\ & & x \otimes y \otimes z & \longmapsto & \Re(xyz) \end{array}$$

   The equivalent reasoning as before (expanding in 1, i, j, k and imposing the conditions) shows that this is a triality.

Observation 13. *It would seem that having a triality is not as easy as having a duality, since if we have the former between V_1, V_2, V_3, say f, then this same mapping induces a duality between V_i and V_j for every $i, j = 1, 2, 3$, $i \neq j$. In particular, $\dim(V_1) = \dim(V_2) = \dim(V_3)$ and Proposition 22 characterizes each and every one of these dualities.*

Consider now a triality between the finite dimensional $\mathbb{R}$ vector spaces V_1, V_2, V_3, say $f : V_1 \otimes V_2 \otimes V_3 \longrightarrow \mathbb{R}$. Define:

$$\begin{array}{rcrcl} \tilde{f} & : & V_1 \otimes V_2 & \longrightarrow & V_3^* \\ & & v_1 \otimes v_2 & \longmapsto & \tilde{f}(v_1 \otimes v_2) \; : \; \begin{array}{rcl} V_3 & \longrightarrow & \mathbb{R} \\ v_3 & \longmapsto & f(v_1 \otimes v_2 \otimes v_3) \end{array} \end{array}$$

the natural way of obtaining elements of V_3^* via f. Choose a non zero $e_1 \in V_1$ and define:

$$\begin{array}{rcrcl} \varphi & : & V_2 & \longrightarrow & V_3^* \\ & & v_2 & \longmapsto & \tilde{f}(e_1 \otimes v_2) \end{array}$$

which is injective because of the non degeneracy conditions over f and since V_2 and V_3^* have the same dimension is thus an isomorphism. Choose a non zero $e_2 \in V_2$ and define:

$$\begin{array}{rcrcl} \psi & : & V_1 & \longrightarrow & V_3^* \\ & & v_1 & \longmapsto & \tilde{f}(v_1 \otimes e_2) \end{array}$$

which is also an isomorphism.

Definition 32. *Let $f : V_1 \otimes V_2 \otimes V_3 \longrightarrow \mathbb{R}$ be a triality over three finite dimensional real vector spaces and $\tilde{f}$, φ, ψ as above. We define $\Phi = \tilde{f} \circ (\psi \otimes \varphi)^{-1}$, the map that makes the following diagram commute:*

$$\begin{tikzcd} V_1 \otimes V_2 \arrow[r, "\tilde{f}"] \arrow[d, "\psi\otimes\varphi"'] & V_3^* \\ V_3^* \otimes V_3^* \arrow[ur, "\Phi"'] & \end{tikzcd}$$

This map is a "product" in V_3 in the sense that it gives him a structure of $\mathbb{R}$ algebra.

Proposition 23. *Let $f : V_1 \otimes V_2 \otimes V_3 \longrightarrow \mathbb{R}$ be a triality over three finite dimensional real vector spaces. Then $\Phi : V_3^* \otimes V_3^* \to V_3^*$ has an identity element and no zero divisors.*

Proof. We claim that $e = \tilde{f}(e_1 \otimes e_2)$ is the identity element. Let $w \in V_3^*$, we have that:

$$\begin{aligned} \Phi(e \otimes w) &= \Phi(\tilde{f}(e_1 \otimes e_2) \otimes w) = \Phi(\psi(e_1) \otimes w) = \Phi(\psi(e_1) \otimes \varphi(\varphi^{-1}(w))) \\ &= \Phi((\psi \otimes \varphi)(e_1 \otimes \varphi^{-1}(w))) = \tilde{f}(e_1 \otimes \varphi^{-1}(w)) = \varphi(\varphi^{-1}(w)) = w, \end{aligned}$$

and in analogous fashion $\Phi(w \otimes e) = w$, and thus e is a bilateral unit.

Suppose that for some $w, \tau \in V_3^*$ we had $\Phi(w \otimes \tau) = 0$. As $\psi \otimes \varphi$ is an isomorphism, there exist $v_1 \in V_1$ and $v_2 \in V_2$ with $\psi(v_1) = w$ and $\varphi(v_2) = \tau$. Then:

$$0 = \Phi(w \otimes \tau) = \Phi((\psi \otimes \varphi)(v_1 \otimes v_2) = \tilde{f}(v_1 \otimes v_2) \Longrightarrow f(v_1 \otimes v_2 \otimes v_3) = 0 \text{ for every } v_3 \in V_3,$$

and thus because of the non degeneracy of f we have that $v_1 = 0$ or $v_2 = 0$, that is, $w = 0$ or $\tau = 0$. □

This results guarantees that V_3^* is in fact a finite dimensional real division algebra. These algebras are extremely rare.

Theorem 8. *Every finite dimensional real division algebra has dimension* 1*,* 2*,* 4 *or* 8*.*

Proof. This is a difficult result to obtain. Heinz Hopf proved in [14] that the dimension must be a power of 2 and Raoul Bott, John Milnor and Michel Kervaire proved independently in [15] and [16] that the dimension must be less than or equal to 8. □

In particular, we may have a triality only in dimension 1, 2, 4 or 8. As seen in Example 7, we already know trialities in the lower dimensions, taking the division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ as our starting point.

Definition 33. *We define the* octonions *as the only real normed division algebra of dimension* 8*, and denote them by* $\mathbb{O}$*.*

This is well defined because as shown in [6] there are only four real normed division algebras, the first three being $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. However, this is certainly not the most useful way to define the octonions. To this effect, we refer to the characterizations in [6] and [1]. We will see that the triality induced by Spin(8) is precisely the same as the triality induced by the octonions in the sense of Example 7.

5.3 The Triality Induced by *Spin*(8)

In this section we will construct a triality that iduces a real division algebra of dimension 8. This will be done taking Spin(8) as our starting point and using the unique property provided by Theorem 7. For this, we will need a few extra results concerning the representations of Spin(8).

Proposition 24. *For the irreducible representations of* Spin(8) *we have that:*

1. *The representation λ is self dual, that is, $\lambda^* = \lambda$.*
2. *The product of the representations $\Delta^{\pm}$ yields another representation having λ as a direct term in a sum decomposition, $\Delta^+ \otimes \Delta^- = \lambda + \Theta$, where Θ is some representation.*
3. *The group* Spin(8) *acts transitively over $\mathbb{S}^7 \times \mathbb{S}^7 \subset \lambda \times \Delta^+$.*

Proof. It can be found in [1]. □

Note that in the last item we make the abuse of notation of interpreting the representations λ, Δ^+, Δ^- as the vector spaces the group acts on, instead of the morphism as we have been doing.

Observation 14. *Note that:*

$$\lambda \otimes \Delta^+ \otimes \Delta^- = \lambda \otimes (\lambda + \Theta) = \lambda \otimes \lambda + \lambda \otimes \Theta,$$

and as $\lambda = \lambda^*$, we in fact have that the first term is the duality $\mu : \lambda \otimes \lambda^* \to \mathbb{R}$. In particular, this first term is linear and equivariant, since for every $g \in \mathrm{Spin}(8)$ and $x, y \in \lambda = \lambda^* = \mathbb{R}^8$ we have that:

$$\begin{aligned}\mu(g \cdot (x \otimes y)) &= \mu((\lambda \otimes \lambda)(g)(x \otimes y)) = \mu(\lambda(g)(x) \otimes \lambda(g)(y)) = \lambda(g)(y) \cdot \lambda(g)(x) \\ &= \lambda(g)(y \cdot x) = \lambda(g)(\mu(x \otimes y)) = g \cdot \mu(x \otimes y).\end{aligned}$$

Definition 34. *For $\mathrm{Spin}(8)$, we define $f : \lambda \otimes \Delta^+ \otimes \Delta^- \to \mathbb{R}$ as the natural restriction on $\lambda \otimes \lambda$. That is, given $v = (v_1 \otimes v_2 \otimes v_3) \in \lambda \otimes \Delta^+ \otimes \Delta^-$ with $v = w = (w_1 \otimes w_2) + (w_3 \otimes w_4) \in \lambda \otimes \lambda + \lambda \otimes \Theta$, we have that $f(v) = \mu(w_1 \otimes w_2) \in \mathbb{R}$.*

We obviously have that f is linear, non zero (since a duality is non zero) and it is $\mathrm{Spin}(8)$ equivariant.

Theorem 9. *The map $f : \lambda \otimes \Delta^+ \otimes \Delta^- \to \mathbb{R}$ is a triality.*

Proof. If any element $x, y, z = 0$, then obviously $f(x \otimes y \otimes z) = 0$.

Suppose there exist non zero $x, y \in \mathbb{R}^8$ with $f(x \otimes y \otimes z) = 0$ for every $z \in \mathbb{R}$, or equivalently $\tilde{f}(x \otimes y) = 0$ as an element of Δ^-. By linearity $\tilde{f}\left((x/\,||x||) \otimes (y/\,||y||)\right) = 0$, and without loss of generality we may take $||x|| = ||y|| = 1$, that is $x, y \in \mathbb{S}^7$. Since $f \neq 0$, there exist $x_0, y_0, z_0 \in \mathbb{R}^8$ for which $f(x_0 \otimes y_0 \otimes z_0) \neq 0$, or equivalently $\tilde{f}(x_0 \otimes y_0) \neq 0$. As before, we may take $||x_0|| = ||y_0|| = ||z_0|| = 1$, or equivalently $x_0, y_0, z_0 \in \mathbb{S}^7$.

Considering Proposition 24, there exists an element $g \in \mathrm{Spin}(8)$ such that $g \cdot x = x_0$ and $g \cdot y = y_0$, and thus:

$$0 = \tilde{f}(x \otimes y) = g \cdot \tilde{f}(x \otimes y) = \tilde{f}(g \cdot x \otimes g \cdot y) = \tilde{f}(x_0 \otimes y_0) \neq 0,$$

which is a contradiction. There are no such elements $x, y \in \mathbb{R}^8$.

Using Theorem 7, for any of the other two possibilities we may now use an outer automorphism to permute the representations in a way that they suit the case we just proved:

$$\begin{array}{rcccccc} f \circ \alpha_{(2,3)} & : & \lambda \otimes \Delta^+ \otimes \Delta^- & \overset{(2,3)}{\longleftrightarrow} & \lambda \otimes \Delta^- \otimes \Delta^+ & \longrightarrow & \mathbb{R} \\ & & x \otimes \quad \otimes z & \longmapsto & x \otimes \quad \otimes z & \longmapsto & f(x \otimes \quad \otimes z), \\ f \circ \alpha_{(1,2,3)} & : & \lambda \otimes \Delta^+ \otimes \Delta^- & \overset{(1,2,3)}{\longleftrightarrow} & \Delta^- \otimes \lambda \otimes \Delta^+ & \longrightarrow & \mathbb{R} \\ & & \otimes y \otimes z & \longmapsto & \otimes y \otimes z & \longmapsto & f(\quad \otimes y \otimes z). \end{array}$$

These compositions are also equivariant, since conjugation by an element of $\mathrm{Spin}(8)$ maintains the equivalence class of the automorphism. We can thus apply the same argument as before, obtaining the desired result. □

This triality induces by Proposition 23 a real division algebra of dimension 8. The very important result that ends this work is:

Theorem 10. *Let* λ, Δ^+, Δ^- *be the irreducible representations of* $\mathrm{Spin}(8)$. *We can identify each and every one of these representations with* $\mathbb{O}$.

Proof. It can be found in [1]. $\square$

Observation 15. *Note that this is done while preserving all the vector space structure, and obviously guarantees that the induced (real) division algebra is* $\mathbb{O}$. *In fact, we have the commutative diagram:*

$$\begin{tikzcd}
\lambda \otimes \Delta^+ \otimes \Delta^- \arrow[rr, "f"] \arrow[d, leftrightarrow, "\cong"] & & \mathbb{R} \quad \textit{with } F(x \otimes y \otimes z) = \Re(xyz) \\
\mathbb{R}^8 \otimes \mathbb{R}^8 \otimes \mathbb{R}^8 \arrow[d, leftrightarrow, "\cong"] & & \\
\mathbb{O} \otimes \mathbb{O} \otimes \mathbb{O} \arrow[uurr, "F"'] & &
\end{tikzcd}$$

6 Conclusion

The results presented in this work cover several major branches of the Mathematics. From the Classical groups to Representation Theory, from the definition of the Clifford algebras to the concept of triality, we saw a glimpse of the modern treatment of a mathematical problem: topology was present in the justification that Spin(8) is a double cover of SO(8), algebra was present in the vector space structures in which the foundations of the division algebras rely and even K-Theory was briefly touched in Bott periodicity.

With our feet on the ground, in this work we were able to prove flabbergasting facts. The relation that lies within the Clifford group Γ_n and the isometries $\mathrm{O}(n)$ is nothing to be sneezed at, making it even more remarkable that we started by generalising a mundane sequence of Real algebras. Not to be understated is the recurrent fact that, as here has been shown, this develops into a double cover of a Classical group, which is very surprising. Moreover, in the appropriate dimension this double cover happens to induce a triality, an extremely restrictive and exigent construction. Astonishment is the only possible reaction.

However, not all is said and done. This summarizes the approach taken by Adams when dealing with triality and Spin(8), but the major part of the very interesting subject of the exceptional groups of Lie type has been avoided. It has captivated my mind and I wish to deepen into this world in future readings.

This work demonstrates how one does not need an extremely deep background to get to the core of a subject. Before writing the volume you are finishing, I thought that every mathematical treatise that was to be worth of this name had to start from basic concepts, be self contained and develop the subject as precisely as was needed. Needless to say, such thing was impossible with the task at hand. It is fundamental to have reached the impasse where the understanding of a matter is not subject to all the nuances of the technical results, this, I consider a personal achievement that alone was worth the effort.

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